

The Two Dimensions of Income Growth

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Abstract

Conventional income growth is undefined for transitions involving losses and can magnify ordinary revenue–cost movement near break-even. I show that income growth is two-dimensional: scale and balance changes reconstruct revenue and cost growth throughout the signed domain. No informative endpoint-only scalar has the same additive, unit-invariant properties through zero; conventional growth omits 19 percent of operating and 28 percent of economic-income transitions. Gaussian revenue–cost dynamics generate heavy-tailed income and income growth. The representation reinterprets growth persistence and valuation and shows that omitted transitions are concentrated in equity-return tails, placing profit, loss, and recovery in one growth system.

JEL classifications: G30, G12, D22, L25.

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1 Introduction

Does income growth persist? Do valuation ratios identify firms with unusually high future growth? Which firm movements accompany extreme shareholder returns? These are central finance questions, but the conventional income growth used to answer them exists only when income is positive at both endpoints [Chan et al., 2003, Sloan and Wang, 2025, Tengulov et al., 2025]. Loss entry, continued loss, and recovery are excluded from the growth object rather than measured with error. Even among positive-income firms, an extreme growth observation need not describe an extreme movement of the firm. These restrictions matter for conclusions about whether growth is persistent or transitory, whether market prices extrapolate it correctly, and whether measured performance captures the firm adjustments most relevant to investors.

A small example reveals both problems. Begin with revenue of 100, cost of 90, and income of 10. If both revenue and cost rise by 10%, income rises to 11, a 10% increase. If revenue instead rises by 10% while cost falls by 10%, income rises to 29: conventional income growth is 190% although neither component movement is extreme. The small initial net magnifies ordinary relative movement in the two large components. If income crosses zero, the growth measure disappears altogether. An improvement from -10 to 10 can come from revenue and cost rising from $(90, 100)$ to $(120, 110)$, or falling from $(990, 1000)$ to $(900, 890)$. The income endpoints are identical; the directions of firm adjustment are opposite.

I show that income growth is two-dimensional. Income is the signed difference between two positive flows, revenue and cost. Its natural growth coordinates are therefore scale change, which records movement common to revenue and cost, and balance change, which records their relative movement. Together they follow the full revenue–cost system through profit, loss entry, continued loss, and recovery. Conventional income growth is a censored nonlinear projection of that system. Scale and balance add no information to revenue and cost. They organize the same information in coordinates that separate magnitude from relative performance and make income growth additive.

The construction begins from familiar size and margin information. Let $R > 0$ be revenue, $C > 0$ economically relevant cost, and $Y = R - C$ signed income. Because revenue and cost are

positive and span orders of magnitude, write $r = \log R$ and $c = \log C$. Define

$$\lambda = \frac{r + c}{2}, \quad \tau = r - c, \quad Y = 2e^\lambda \sinh(\tau/2). \quad (1)$$

Scale λ is their common log magnitude. Balance τ is their log ratio: positive in profit, zero at break-even, and negative in loss. The hyperbolic form is the exact algebra of a difference between two positive exponential quantities, much as polar coordinates are the natural representation of circular geometry. It maps every revenue–cost pair to one point, and every point back to one revenue–cost pair. Income is then a contour in this plane rather than a complete description of the position.

The same rotation gives growth:

$$\begin{aligned} \Delta\lambda &= \frac{\Delta r + \Delta c}{2}, & \Delta\tau &= \Delta r - \Delta c, \\ \Delta \log |Y| &= \Delta\lambda + \log \left| \frac{\sinh(\tau_{t+1}/2)}{\sinh(\tau_t/2)} \right| & & \text{for same-sign endpoints.} \end{aligned} \quad (2)$$

The first line reconstructs revenue and cost growth exactly. The second shows what their scalar income growth projection combines. Near break-even, a small balance change acts on large revenue and cost flows but is divided by a small initial net. This is *income growth amplification*. The other consequence is *censoring*: when an endpoint is non-positive, conventional log income growth is undefined even though scale and balance continue to move.

The problem is structural. Scale and balance changes are finite through zero, additive over time, unit invariant, and sufficient to recover both component growth rates. I show that no informative endpoint-only scalar growth measure can satisfy the corresponding additivity, unit-invariance, and continuity requirements on the signed domain. The formal results explain why the constructive answer must look behind the net income endpoint. An all-purpose intrinsic endpoint index is unavailable, while the observed revenue and cost flows supply the minimum-dimensional growth object.

I study these ideas in a 1970–2019 CRSP/Compustat panel of U.S. public operating firms. The analysis uses three ordinary definitions of income. Gross income subtracts COGS from sales; operating income also subtracts SG&A; economic income measures resources retained as net investment

plus resources distributed to the maintained claimant set, with cost defined as the complement to sales. Operating income is primary. Each definition has the same revenue-minus-cost form and its own eligibility, allowing the paper to distinguish the logic of signed income from the choice of cost or income definition.

The empirical facts make both distortions material. Conventional growth is undefined for 18.8 percent of otherwise observable annual operating-income transitions and 27.5 percent at the economic-income boundary. Signed income has substantial mass on both sides of zero, and firms occupy broad two-dimensional position and growth planes. Their marginal distributions are strongly heavy-tailed.¹ Among positive-to-positive transitions, 9–11 percent of income growth observations exceed three IQR-implied standard deviations, roughly 30 times the Normal rate. These extremes are concentrated near break-even, and the nonlinear balance term dominates them at the operating and economic boundaries. Marginal difference-of-lognormals (DLN) fits closely track both signed income and positive-income growth. The distributional facts therefore line up with the geometry: the tails are largest where the scalar growth measure is most sensitive to balance movement.

Scale and balance are also economically real coordinates rather than renamed descriptive ratios. Scale is extraordinarily persistent: its same-firm correlation is about 0.99 after one year and 0.91 after ten. Balance is persistent but substantially more mobile. Conditional on one another, the two coordinates separately organize subsequent loss, adverse exit, asset, PPE, and employment growth, investment, distributions, and debt. Their signs and magnitudes often differ. Adverse performance and liquidation exits rise sharply at low or negative balance within scale groups, whereas M&A is largely unaffected. Scale and balance behave as distinct observable firm states. Each coordinate continues to organize a different part of the firm’s future after the other coordinate and familiar controls are held fixed.

The income map then provides a simple and transparent mechanism for the distributional evidence. I estimate a scale–balance law, impose Gaussian innovations, and map the simulated transitions back into signed income and conventional income growth. The coordinate inputs are Gaussian by construction, yet the income outputs are strongly heavy-tailed. At the operating boundary, for example, 5.40 percent of simulated positive-income growth exceeds three IQR-implied standard

¹I use *heavy-tailed* operationally when a distribution has significantly more tail mass than its Normal benchmark; examples include kurtosis significantly above three or two-sided three- $\tilde{\sigma}$ incidence significantly above 0.27 percent.

deviations, compared with a Normal rate of 0.27 percent. Empirical coordinate innovations are themselves heavy-tailed and feed further extreme movement through the same map. The Gaussian experiment establishes that the map is sufficient to generate substantial tails; the data contain both income growth amplification and extreme coordinate movement. This separates two forces that a scalar income-growth distribution alone cannot distinguish: the law governing firm movement and the nonlinear map from that movement to a signed net outcome.

The first finance application asks what familiar predictors forecast once the outcome is spanned into this growth space. I first compare conventional income growth with scale and balance change on the same positive-to-positive firms, then restore every observed income-sign transition. Coefficients that appear large in conventional growth can correspond to ordinary coordinate movement, and their scale direction can be reversed by balance. At the economic-income boundary, for example, net debt predicts 7.96 log points of conventional income growth while predicting a 1.10-point scale decrease on the same firms. In the uncensored sample, it predicts a 1.99-point scale decrease and a 1.71-point half-balance increase, which nearly cancel in revenue growth. Restoring the censored firms changes some predictor relationships in magnitude and others in sign. The economically relevant comparison asks what movement a familiar coefficient describes and which firms had to be omitted for that coefficient to exist.

The second application changes the interpretation of income-growth persistence. Conventional positive-to-positive income growth mean reverts. On those identical observations, scale change is positively persistent while balance change reverses. The same pattern remains when loss entry, continued loss, and recovery are restored. Familiar income-growth mean reversion is therefore not general reversal of firm growth. It is balance reversal alongside persistent scale movement. The distinction is sharpest after loss entry, where the scale path reverses even though scale growth remains persistent after a positive-to-positive transition. A single persistence coefficient averages economically different adjustment histories.

The valuation results make the distinction economically large. High-minus-low q firms have only 2.19 annualized log points more five-year operating income growth in the conventional positive-to-positive sample. The same firms have 8.09 points more operating-scale increase and a -5.90 -point balance contribution. Holding the ex-ante portfolios fixed and including all the censored endpoints raises the operating-scale spread to 9.19 points. High valuation does predict substantial subsequent

operating-scale increase; deteriorating balance conceals much of it in bottom-line growth. The result also appears across economic income and an EPS measure.

The final application locates contemporaneous equity-return tails in the growth plane. Positive-to-positive transitions are 81.0 percent of matched operating observations but only 33.2 percent of lower-tail mass; at the economic boundary the shares are 72.6 and 20.2 percent. Loss entry and continued loss account for most of the missing lower tail, while recovery is disproportionately represented in the upper tail. Extreme scale or balance movement constitutes about one-fifth of observations but roughly three-fifths of lower-tail mass and 45 percent of upper-tail mass. Tail incidence is near the Gaussian benchmark at the center of the growth plane and rises sharply in particular directions away from it. Conventional income growth analysis omits the firm histories most represented when shareholder outcomes are extreme.

These results connect finance questions that have been treated separately. Research on income-growth persistence and valuation studies growth where the scalar is measurable or moves to positive objects such as sales and EBITDA [Lakonishok et al., 1994, Chan et al., 2003, Sloan and Wang, 2025, Tengulov et al., 2025]. Research on losses documents a distinct persistence, valuation, restructuring, financing, and exit regime [Hayn, 1995, Joos and Plesko, 2005, John et al., 1992, Ofek, 1993, Denis and Kruse, 2000, Anderson et al., 2003, Denis and McKeon, 2021]. The paper places profit, loss entry, continued loss, and recovery in one continuous growth system. The apparent separation partly reflects a growth measure that cannot traverse their boundary. This changes the interpretation of both literatures: positive-domain results describe a selected region of the system, while loss observations are transitions in that same system rather than an empirical category requiring a separate concept of growth.

Two adjacent literatures clarify the contribution. Financial-statement analysis looks behind net performance because margins, turnover, size, and recurring components have distinct forecasting and valuation content [Fairfield and Yohn, 2001, Soliman, 2008, Frank and Gao, 2025]; scale and balance provide an exact additive basis that continues through zero. On heavy-tailed firm dynamics, Jaimovich et al. [2025] show that a non-Gaussian transition law for residualized log revenue affects value, exit, and policy in a Hopenhayn-style model. The present paper identifies a different mechanism: a signed net outcome can have heavy-tailed levels and growth even under a Gaussian coordinate law. Transition-law tails and income-map amplification are distinct forces that operate

together in the data. The first determines how firms move through the plane; the second determines how a given movement appears in income. Separating them matters for any model asked to match income-growth tails or evaluate decisions that depend nonlinearly on income.

The formal foundation extends positive-domain and signed-domain growth measurement [Törnqvist et al., 1985, Studený and Mezník, 2013, Alexeev and Sokolov, 2014, Alekseev and Sokolov, 2021] and complements the nonnegative-outcome impossibility in Chen and Roth [2024]. The constructive answer also supplies an observable target for dynamic corporate-finance and firm-dynamics models [Hennessy and Whited, 2005, 2007, Strebulaev and Whited, 2012, Jovanovic, 1982, Hopenhayn, 1992, Luttmer, 2007]. Any model that generates revenue and cost traces a path in the scale–balance plane. This paper establishes and measures observable position, its movement, and their finance consequences. The target is deliberately stated in observables: competing production, adjustment-cost, financing, and exit mechanisms can be compared by the restrictions they place on the same scale–balance plane.

The argument is a connected chain. Signed income has a natural two-dimensional representation; its movement has an additive through-zero growth vector; conventional scalar growth both amplifies and censors that movement; and correcting those distortions changes substantive finance evidence.

2 Signed Income and Income Growth

Signed income (profit or loss, henceforth simply income) is the net outcome of two firm flows: revenue minus cost. Firms with the same income amount can have revenue–cost systems of very different magnitudes, and the same income improvement (in level or percent) can accompany an increase or decrease in scale of operations. This compression implies that measuring income growth requires recovering the two-dimensional observable positions of income and of income growth.

This section constructs the required objects. Scale and balance first provide a natural and exact two-dimensional representation of signed income. Their changes then measure common and relative revenue–cost movement throughout the income domain. It also establishes four theoretical results: (i) income growth cannot be represented by a one-dimensional scalar that conforms with common axioms; (ii) taken together, changes in scale and balance conform with the growth axioms;

- (iii) this change vector is the canonical growth measure under mild symmetry assumptions; and
- (iv) the minimal dimension to losslessly represent the income or income growth system is two.

Finally, the section maps native movement in scale–balance space back into income and shows two additional results. First, the common measure of income growth (difference in log income for positive-to-positive movement) includes a non-linear amplification mechanism. The amplification causes small movements in scale–balance space to appear as large income growth movements. The second establishes that even under the simplifying assumption of Gaussian movements in scale–balance space, the amplification mechanism causes traditionally-measured income growth to have significant heavy-tails. This introduces heavy-tailed growth into the firm optimization problem from basic accounting principles and using only Normal underlying shocks.

2.1 Signed income and the scale–balance space

Let revenue $R > 0$, cost $C > 0$, and signed income $Y = R - C$, and let lower-cases denote log values as usual. Revenue and cost are always-positive and span several orders of magnitude, so are naturally expressed in logs. Define

$$\lambda = \frac{r + c}{2}, \quad \tau = r - c. \quad (3)$$

to be the *scale* and *balance* of the revenue–cost system. Throughout the paper, “observable position” refers to position in (λ, τ) space.

Proposition 1 (Scale–balance representation). *The map $(R, C) \mapsto (\lambda, \tau)$ is a smooth bijection from $\mathbb{R}_{++}^2$ to $\mathbb{R}^2$,² with inverse*

$$R = e^{\lambda + \tau/2}, \quad C = e^{\lambda - \tau/2}. \quad (4)$$

Signed income is recovered exactly as

$$Y = 2e^\lambda \sinh(\tau/2). \quad (5)$$

Scale is the log geometric mean of revenue and cost, or equivalently their common log magnitude.

² $\mathbb{R}_{++}^2$ is the positive-positive quadrant of $\mathbb{R}^2$.

Balance is their log ratio and records their proportional difference, or “wedge”. It determines the sign of income: profit, break-even, and loss correspond to $\tau > 0$, $\tau = 0$, and $\tau < 0$. Familiar profitability and productivity margins are monotone normalizations of the same relative coordinate:

$$\tau = \log \left(1 + \frac{Y}{C} \right) = -\log \left(1 - \frac{Y}{R} \right), \quad \frac{R - C}{R + C} = \tanh(\tau/2).$$

The raw ingredients are familiar size and margin or revenue and cost information. The $\lambda\tau$ rotation places both in symmetric log units, inverts exactly to the two original flows, and generates the additive growth basis below. It does not create or contain more information than revenue and cost or size and margin.

Nevertheless, the choice of the definition of scale and balance, and the emergence of hyperbolic trigonometric functions is not arbitrary. The hyperbolic form is the natural algebra of a difference of positive exponential quantities:

$$e^r - e^c = 2e^{(r+c)/2} \sinh((r - c)/2).$$

Hyperbolic geometry enters because income, the economic object of interest, is literally a difference of exponentials — the construct hyperbolic geometry was designed to represent. The mapping $(R, C) \mapsto (\lambda, \tau)$ can perhaps best be understood as the hyperbolic equivalent of the familiar mapping from cartesian to polar coordinates in circular geometry, typically written as $(X, Y) \mapsto (r, \theta)$.

Figure 1 supplies the paper’s visual grammar. Panel A separates income from position: iso-income curves cross scale, and break-even is the line $\tau = 0$. Panel B separates common movement from relative rebalancing: its diagonals partition all possible signs of revenue and cost growth. The figure also poses the next question: If signed income can pass through zero while the gross flows remain positive, can one scalar growth index follow it without losing the standard disciplines of growth measurement? We turn to this question next.

2.2 Why income growth cannot be an intrinsic scalar

A growth measure should aggregate adjacent annual changes into the same beginning-to-end movement, say the same thing in dollars and thousands of dollars, and vary continuously as its endpoints

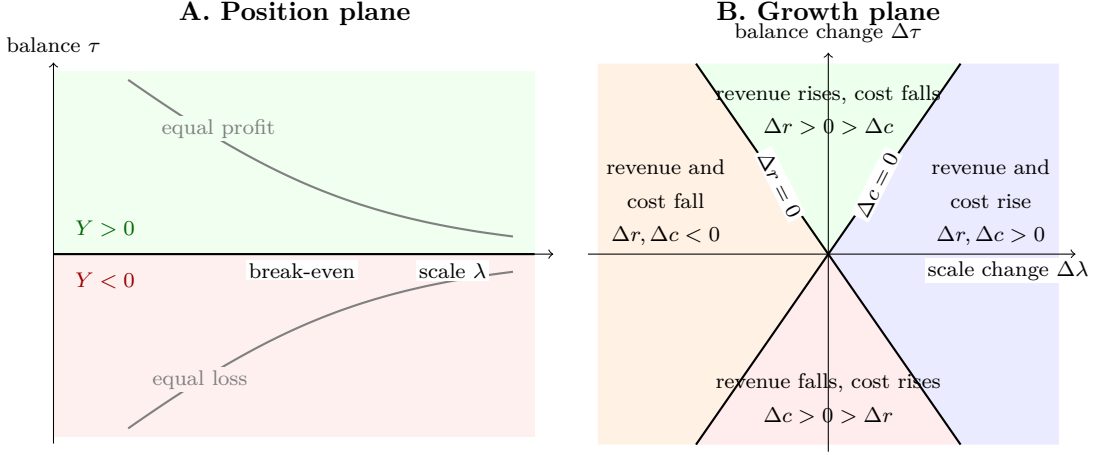


Figure 1. **The position and growth planes behind signed income.** Panel A shows the observable position under $Y = 2e^\lambda \sinh(\tau/2)$. Gray curves hold positive and negative signed income fixed; $\tau = 0$ is break-even. Panel B shows native movement. Since $\Delta r = \Delta\lambda + \Delta\tau/2$ and $\Delta c = \Delta\lambda - \Delta\tau/2$, the diagonal lines mark zero revenue and zero cost growth and partition the plane by the signs of component movement.

Alt text: Panel A places balance vertically and scale horizontally. Equal-profit and equal-loss curves approach the horizontal break-even line as scale rises. Panel B partitions scale and balance changes into regions where revenue and cost both rise, both fall, or move in opposite directions.

move. All three are accepted basic axioms in the growth measurement literature [Törnqvist et al., 1985, Studený and Mezník, 2013, Alexeev and Sokolov, 2014, Alekseev and Sokolov, 2021]. The results here extend the positive-domain growth-measurement logic of Törnqvist et al. [1985], Studený and Mezník [2013] and the signed-domain analysis in Alexeev and Sokolov [2014], Alekseev and Sokolov [2021]. The closest paper in that literature, Chen and Roth [2024], establishes a related impossibility for transformation-based treatment effects with nonnegative outcomes. While they deal with growth in semi-positive values (i.e., the positive range and 0), the problem here places zero inside a larger signed domain. The results below regarding $(\Delta\lambda, \Delta\tau)$ sufficiency provide a constructive counterpart to their scalar impossibility.

Let $G(x, y)$ be some measure of movement from signed-income endpoint x to endpoint y . Call it *intrinsic and additive* when it satisfies the three restrictions:

- R1.** *Temporal additivity:* $G(x, z) = G(x, y) + G(y, z)$;
- R2.** *Intrinsic unit invariance:* $G(ax, ay) = G(x, y)$ for every $a > 0$; and
- R3.** *Continuity:* the measure varies continuously with its endpoints.

Additivity already implies zero movement from an endpoint to itself and reversal when endpoints

are exchanged. Normalize $G(1, e) = 1$ to express growth in natural log units. This normalization fixes units and carries no economic restriction. On positive support, these requirements directly select log-point growth as the default measure of growth.

Lemma 1 (Positive-domain benchmark). *If $G : \mathbb{R}_{++}^2 \rightarrow \mathbb{R}$ is temporally additive, intrinsically unit invariant, continuous, and normalized by $G(1, e) = 1$, then*

$$G(x, y) = \log y - \log x.$$

The economic content is simple and well-known: on a positive ratio scale, temporal aggregation, unit invariance, continuity, and log-point normalization leave no alternative to log growth. Signed income introduces an interior zero, leaves logs undefined or complex-valued, and hence leads to the central theoretical impossibility outcome.

Theorem 1 (Impossibility of intrinsic scalar growth on the signed domain). *There is no normalized, informative endpoint measure on $\mathbb{R}$ that satisfies requirements R1–R3. More precisely, if $G : \mathbb{R}^2 \rightarrow \mathbb{R}$ is temporally additive, intrinsically unit invariant, and continuous at 0, then*

$$G(x, y) = 0 \quad \text{for every } x, y \in \mathbb{R}.$$

The proof (in the appendix) explains why zero is decisive. Additivity makes every endpoint measure a potential difference, $G(x, y) = h(y) - h(x)$. Because zero is fixed by proportional rescaling, unit invariance makes h constant along each signed ray. Continuity at zero equates both ray constants with the value at zero, leaving nothing for the scalar to measure. The argument is distribution-free and leaves directionality and differentiability unrestricted.

2.3 Native two-dimensional growth

Revenue and cost remain positive as their difference crosses zero, so their log changes remain defined. Let $\Delta r = \Delta \log R$ and $\Delta c = \Delta \log C$ be the growth in revenue and growth in costs. The scale-balance rotation then provides the through-zero growth object.

Proposition 2 (Scale–balance growth). *Define*

$$\Delta\lambda = \frac{\Delta r + \Delta c}{2}, \quad \Delta\tau = \Delta r - \Delta c. \quad (6)$$

The vector $(\Delta\lambda, \Delta\tau)$ is intrinsically unit invariant, finite through an income zero or sign crossing whenever $R, C > 0$, and additive over time. It reconstructs component growth exactly:

$$\Delta r = \Delta\lambda + \frac{1}{2}\Delta\tau, \quad \Delta c = \Delta\lambda - \frac{1}{2}\Delta\tau. \quad (7)$$

A positive $\Delta\lambda$ records an operating-scale increase; a negative value records an operating-scale decrease. A positive $\Delta\tau$ means that revenue grows relative to cost and, for a loss firm, that balance moves toward break-even. A common change of currency units shifts every λ level by the same constant, but the constant cancels from $\Delta\lambda$; balance and balance change are already unit invariant. Both coordinates telescope:

$$\sum_s \Delta\lambda_s = \lambda_T - \lambda_0, \quad \sum_s \Delta\tau_s = \tau_T - \tau_0.$$

The beginning-to-end movement is therefore independent of the frequency at which the interval is partitioned.

Many invertible rotations of $(\Delta r, \Delta c)$ preserve information. The economically useful basis $(\Delta\lambda, \Delta\tau)$ separates movement common to both gross flows from movement that reverses when their roles are exchanged. The exchange requirement imposes coordinate parity while allowing revenue and cost to have different adjustment technologies and economic meanings. Additivity and this common–relative symmetry select the scale–balance rotation as the canonical one.

Proposition 3 (Canonical common–relative basis). *Let $\hat{G} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a continuous, lossless representation of component log growth. Require path additivity,*

$$\hat{G}(u + v) = \hat{G}(u) + \hat{G}(v) \quad \text{for all } u, v \in \mathbb{R}^2. \quad (8)$$

Require common–relative symmetry, such that an exchange of the two component roles leave the first coordinate unchanged and reverse the second. Finally, normalize common proportional growth

to be recorded as $(g, 0)$ and the units to natural logs. Then

$$\hat{G}(\Delta r, \Delta c) = \left(\frac{\Delta r + \Delta c}{2}, (\Delta r - \Delta c) \right) = (\Delta \lambda, \Delta \tau) \quad (9)$$

“Canonical” applies within the stated continuous, additive, lossless, common–relative class. The next result establishes the minimum dimension needed to retain both movements continuously.

Proposition 4 (Minimum continuous dimension). *On any open two-dimensional set of flow positions or component-growth movements, a continuous lossless representation must have dimension at least two. In particular, no continuous scalar representation is lossless on that set.*

Propositions 2–4 establish the constructive content of the representation. Two coordinates reconstruct revenue and cost flow movement exactly, open two-dimensional support requires two continuous coordinates, and common–relative symmetry selects the scale–balance basis.

Financial-statement analysis already looks behind net outcomes. DuPont analysis separates margins from turnover or size because the components have distinct forecasting and valuation content [Fairfield and Yohn, 2001, Soliman, 2008]; recent work likewise studies size and profitability jointly [Frank and Gao, 2025]. This paper’s additional object is the exact common–relative representation of two positive gross flows and the additive growth vector that remains defined when their signed difference crosses zero. The next subsection maps that movement back into signed income.

2.4 From native movement to income growth amplification

The observable position (λ, τ) records the firm’s full income position and the native movement in this space $(\Delta \lambda, \Delta \tau)$ records how income moves. Conditional on the initial observable position (λ_t, τ_t) , the exact next-period income is

$$Y_{t+1} = 2e^{\lambda_t + \Delta \lambda} \sinh\left(\frac{\tau_t + \Delta \tau}{2}\right), \quad (10)$$

and for a positive-to-positive income move $(Y_t, Y_{t+1} > 0)$ for which traditional income growth is defined,

$$\Delta \log Y = \Delta y = \Delta \lambda + \log \frac{\sinh((\tau_t + \Delta \tau)/2)}{\sinh(\tau_t/2)}. \quad (11)$$

The same native movement can therefore produce different income changes at different τ_t origins. An observed income growth measure compresses and non-linearly transforms the native movement. The native vector records how the firm moves. Signed income records the net amount generated by that movement at a particular origin.

The local income map makes the origin dependence especially transparent:

$$dY = (R - C) d\lambda + \frac{R + C}{2} d\tau = Y d\lambda + \frac{R + C}{2} d\tau. \quad (12)$$

Common movement $d\lambda$ is multiplied by net income Y , or the difference of revenue and cost, while balance movement is multiplied by the average of revenue and cost. Close to break-even the first-order loading on common movement vanishes, while balance movement acts on a typically large base.

Proposition 5 (Near-break-even income growth amplification). *Fix a finite scale movement $\Delta\lambda$ and a nonzero balance movement $\Delta\tau$. Along any positive income sequence with $\tau \rightarrow 0$,*

$$|\Delta y| \rightarrow \infty. \quad (13)$$

The analogous percentage response $(Y_{t+1} - Y_t)/Y_t$ also diverges in absolute value. By contrast, the native movement $(\Delta\lambda, \Delta\tau)$ remains fixed and finite.

The proposition identifies income growth amplification. Begin with revenue 100 and cost 90. Revenue growth of ten percent and cost contraction of ten percent raise income from 10 to 29: relatively ordinary component movements produce a 190-percent income response. At initial revenue 1,000 and cost 990, the same ten-dollar income lies much closer to break-even in balance space; opposing ten-percent component movements take income to 209 or -189 , a growth of 1,990% or an undefined percentage decline. Initial scale, initial balance, and native movement jointly determine the scalar response.

The proposition is deterministic; a position distribution and movement law turn it into a distributional mechanism. With position mass near break-even and nondegenerate balance movement, ordinary coordinate changes can create substantial scalar tails. If native movement is itself heavy-tailed it can provide a second source of income growth tails. The empirical design in Section

3 separates income growth amplification under a Gaussian input benchmark from the additional native tails in empirical coordinate innovations.

Corollary 1 (Gaussian input benchmark). *Suppose $x_t = (\lambda_t, \tau_t)'$ follows*

$$x_{t+1} = \mu + A(x_t - \mu) + \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim N(0, \Sigma), \quad \rho(A) < 1.$$

With the innovations ε_{t+1} independent over time and independent of x_t . At the invariant distribution, $x_t \sim N(\mu, \Omega)$, where $\Omega = A\Omega A' + \Sigma$, and the full position–movement vector $(x_t, \Delta x_{t+1})$ is jointly Gaussian. Nevertheless,

$$Y_t = e^{\lambda_t + \tau_t/2} - e^{\lambda_t - \tau_t/2}$$

and income is a difference of correlated lognormal variables. Conventional income growth,

$$\Delta y = \Delta \lambda + \log \frac{\sinh((\tau_t + \Delta \tau)/2)}{\sinh(\tau_t/2)}$$

when defined, is a sum of the Normal component $\Delta \lambda$ and the non-linear and amplified second component, leading to large deviations from Normality in Δy . Gaussianity of positions and native movements therefore places no Gaussian restriction on signed-income levels, changes, or scalar income-growth responses.

The corollary provides two benchmark observations. In the simple benchmark case where $(\lambda_t, \tau_t, \Delta \lambda_t, \Delta \tau_t)$ are all jointly Gaussian, income distributes as the difference-of-log-Normals (henceforth DLN), and conventional income growth exhibits heavy-tails as a result of the amplification at low initial τ_t values. This is an economically-transparent and simple mechanism for the emergence of heavy-tails in firm dynamics, even when the underlying dynamics are Gaussian. The implied empirical test is direct: impose Gaussian coordinate positions and movements, then measure the tails and distributions of signed income and positive-to-positive conventional income growth.

3 Position, Movement, and Income Growth in Firm Data

Taking the theoretical constructs to the data, this section shows that Firms occupy both dimensions of the scale–balance space, both are economically active and behave like we would expect

persistent firm states to behave, and the two have distinct dynamics and finance implications and associations. Furthermore, conventional income growth omits material observable histories and has heavy systematic tails that can be reproduced using a Gaussian benchmark.

The section is descriptive, and its main goal is to establish the position space and growth space of Figure 1 as economically meaningful and organizing firm outcomes. Finance conclusions or intuitions changed by the theory are discussed in Section 4.

3.1 Data, income boundaries, and the empirical domain

Data are predominantly from the annual CRSP/Compustat panel covering U.S. public operating firms during the 50-year period 1970 through 2019. I exclude financial firms, utilities, and government enterprises and require data availability for computing the positive revenue and cost inputs needed to locate each position. The unit of observation is a firm fiscal year. Static dollar positions are shares of nominal GDP expressed in units of the 2019 economy.³ Appendix B reports the sample screens, precise deflation mechanism, and maps definitions to CRSP/Compustat observables.

Defining income in the data requires defining empirical revenue and cost. Revenue is relatively straightforward: it is just firm sales. Costs are a more delicate matter. The economically relevant cost definition changes with the question. *Gross income* subtracts COGS from sales. Operating income subtracts both COGS and SG&A and provides the paper’s primary income definition. I also define Economic income using the firm’s core sources-and-uses identity. It equals the sum of net capital retained in the firm and net capital disbursed to the firm’s stakeholders — equity and debt holders, with economic cost defined as sales less that income. This definition is useful as a starting point for finance models and analysis, as it captures precisely the surplus that later investment, financing, and disbursement decisions — the three core finance decisions of the firm — rely on. Its cost definition is hence expansive and implicitly includes “special / one-time” expenses, encumberments, and adjustments. The three income definitions are maintained assumptions on the same identity and each has independent eligibility.

For $j \in \{G, O, E\}$, let C^j be the cost at the gross, operating, or economic-income boundary and

³I.e., for any dollar value X_t , I calculate and use $\tilde{X}_t = (X_t/NGDP_t) \cdot NGDP_{2019}$. This removes both inflation and the secular growth trend and makes the economy stationary. It is the equivalent of taking "year fixed effects" from each dollar value.

set

$$Y^j = R - C^j, \quad \lambda^j = \frac{\log R + \log C^j}{2}, \quad \tau^j = \log(R/C^j). \quad (14)$$

Table 1 presents the descriptive statistics of the three income layers, showing both common characteristics and different features between them. Significant portions of the samples register losses or lack a traditional growth measure because one or both yearly incomes is negative. Median balance τ shrinks rapidly from 0.391 to 0.064 as we move down the income ladder, while scale and the dispersions of balance, scale, and their innovations are roughly similar. Scales are highly correlated between income layers, while balances less so.

Panel B shows that within the positive-to-positive income domain, both median income growth and its dispersion increase as we move down the income ladder. Extreme growth occurs in 9-10% of observations, about 30 times higher than a Normal growth distribution will predict. With the growth decomposition of Equation 11, the non-linear balance term dominates the $\Delta\lambda$ term in 59.5, 92.4, and 98.6 percent of those extreme-growth events, indicating that a considerable amount of conventional measured income growth tails is in fact mere amplification. Within every boundary, extreme growth is most frequent among firms beginning closest to break-even, as amplification would predict.

Figure 2 then gives a single view of the empirical system. Panels a,b show that operating income has substantial mass on both sides of zero, appears to be log-Normally distributed in both the positive and negative directions, and a DLN distribution closely tracks its marginal distribution. Those same observations fill broad two-dimensional position and movement planes, as panels c,d show. Two striking facts in panel c are the near conditional independence of scale and balance, characterized by a nearly flat “ridge”, and the visibly “empty” region at high scale and negative balance, which would be a natural outcome of exit or retrenchment of firms losing large sums of money. Panel e shows that positive-to-positive income growth Δy also exhibits excellent visual fit to the heavy-tailed DLN distribution, and Panel f ascertains that the contribution of the non-linear balance term to Δy decreases as the balance increases, as expected. Because Δy^O is defined only for positive-to-positive transitions, the $\Delta y^O - \Delta \lambda^O < 0$ range in Panel f is censored for firms entering loss.

Table 1. **Income boundaries, native coordinates, and positive-income growth.** Every boundary has the form $Y^j = R - C^j$. Gross income subtracts COGS from sales; operating income also subtracts SG&A. Economic income is net investment retained by the firm plus distributions to its stakeholders, with implied cost equal to sales minus that income. Each boundary uses its independently eligible sample of one-year transitions. Panel A reports origin positions and origin loss rates alongside changes over those same transitions. Panel B considers Δy and so requires positive income at both endpoints. Define $\tilde{\sigma}_y = \text{IQR}(\Delta y)/1.34898$, the robust standard deviation of Δy . An extreme change is defined as $|\Delta y| > 3\tilde{\sigma}_y$.

	Gross income	Operating income	Economic income
<i>Panel A: Income boundaries and native coordinates</i>			
One-year transitions	141,265	141,979	139,361
Origin loss rate, $\Pr(\tau_t < 0)$	3.1%	13.9%	18.5%
Ordinary log growth undefined	4.0%	18.8%	27.5%
Median(λ_t)	6.214	6.369	6.417
Median(τ_t)	0.391	0.109	0.064
SD(λ_t)	2.100	2.056	2.052
SD(τ_t)	0.515	0.377	0.371
SD($\Delta\lambda$)	0.293	0.268	0.282
SD($\Delta\tau$)	0.262	0.213	0.321
corr(λ_t, τ_t)	-0.104	0.260	0.203
corr($\Delta\lambda, \Delta\tau$)	-0.134	0.222	-0.190
corr(λ_t^j, λ_t^O)	0.996	1.000	0.999
corr(τ_t^j, τ_t^O)	0.615	1.000	0.840
<i>Panel B: Positive-income growth and its balance mechanism</i>			
Positive-income transitions	135,552	115,229	100,983
Median(Δy)	0.016	0.033	0.041
Median($ \Delta y $)	0.121	0.189	0.308
Normal-equivalent dispersion, $\tilde{\sigma}_y$	0.178	0.274	0.452
Extreme growth, $\Pr(\Delta y > 3\tilde{\sigma}_y)$	9.1%	10.6%	8.7%
Expected Normal rate	0.28%	0.29%	0.28%
Balance term dominates in extreme-growth events	59.5%	92.4%	98.6%
$\Pr(\Delta y > 3\tilde{\sigma}_y \mid 0 < \tau_t \leq 0.05)$	52.2%	31.6%	15.6%
$\Pr(\Delta y > 3\tilde{\sigma}_y \mid 0.05 < \tau_t \leq 0.10)$	24.6%	10.8%	5.7%
$\Pr(\Delta y > 3\tilde{\sigma}_y \mid 0.10 < \tau_t \leq 0.15)$	16.0%	7.0%	5.5%

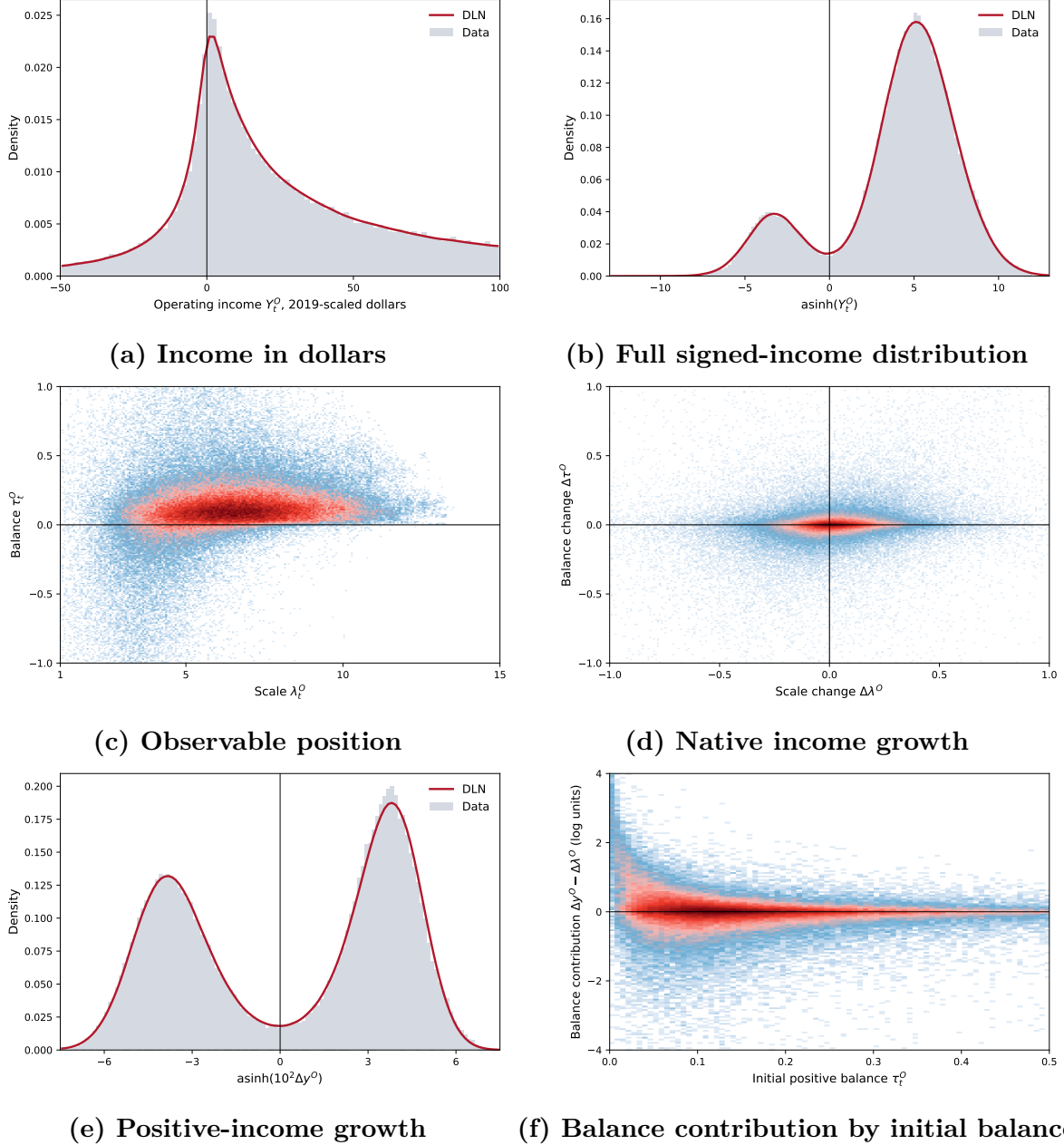


Figure 2. Operating income: distribution, position, and growth. Panels A and B compare the empirical operating-income density with a marginal DLN fit estimated on the same 163,547 nonzero observations. Panel A truncates only the displayed dollar range; Panel B shows the full signed sample under asinh. Panels C and D plot exact observable position (λ_t^O, τ_t^O) and native income growth ($\Delta \lambda^O, \Delta \tau^O$). Panel E restricts attention to the 115,229 transitions with $\tau_t^O, \tau_{t+1}^O > 0$, defines $\Delta y^O = \log Y_{t+1}^O - \log Y_t^O$, expresses it in log points as $10^2 \Delta y^O$, and overlays its marginal DLN fit. Panel F presents the heatmap of balance contribution $\Delta y^O - \Delta \lambda^O = \log\{\sinh(\tau_{t+1}^O/2)/\sinh(\tau_t^O/2)\}$ against initial positive balance.

Alt text: Panels A and B show an asymmetric two-sided operating-income distribution closely tracked by the difference-of-lognormals fit. Panel C shows a dense, nearly horizontal positive-balance ridge and few high-scale losses. Panel D centers native changes near zero. Panel E shows heavy-tailed positive-income growth. Panel F shows balance contributions dispersing widely near break-even and narrowing as initial balance rises.

3.2 Economic content of scale and balance

Scale and balance carry different and meaningful economic content, and behave as one would expect separate firm states to behave. Scale is highly persistent, even at a 10-year horizon. Balance, while also highly persistent initially, has lower persistence than scale and it diminishes more quickly over time. The two have significant, different, and sometime opposing impact on future firm outcomes such as exit, real expansion, and financial policies. Finally, a Gaussian benchmark ascertains that jointly-Normal position and movement can nevertheless generate heavy-tailed distributions similar to those observed in the data.

Table 2 reports exact-horizon same-firm correlations at one, three, five, and ten years. A pair enters only when the firm has an eligible position at both endpoints. The entries are hence observed-survivor correlations. Panel B then asks whether each coordinate organizes subsequent operating, real, and financial outcomes conditional on the other coordinate and standard controls.

Scale correlations begin near 0.99 at every boundary and remain about 0.90 after ten years. Balance is also very persistent initially, but materially more mobile: its ten-year correlation is 0.578 at the gross boundary, 0.464 at the operating boundary, and 0.279 at the economic boundary. The distinction resembles the familiar finance contrast between persistent firm size and more mobile profitability [Fama and French, 2000, Frank and Gao, 2025], but here both are exact coordinates of one revenue–cost position.

The conditional outcome gradients give the persistence evidence its economic meaning. The magnitudes are percentage-point changes in an outcome from a “typical” scale/balance movement. Current balance organizes next-year loss, classified adverse exit, real expansion, and financing choices after current scale and its square are controlled for. Current scale retains its own gradients conditional on balance. Each observable position coordinate is associated with distinct parts of the firm’s subsequent path. For both coordinate dynamics and financial policy outcomes, balance changes are more statistically and economically significant than scale changes, often with opposite signs.

Figure 3 makes the exit distinction visible over the full position plane. It contrasts adverse performance/liquidation delistings with M&A, an alternative route out of the public panel. The two exit types provide a direct check on whether low balance specifically organizes adverse resolution

Table 2. **Scale and balance are distinct firm states.** Panel A reports Pearson same-firm correlations at exact horizons among firms observed at both endpoints. Its N range is the minimum and maximum number of firm pairs across the gross, operating, and economic boundaries. Panel B reports average fitted changes in event probabilities from raising each current operating coordinate by the IQR of its one-year movement, listed in parenthesis. N is firm-year observations; baseline event rates in percentage point terms are listed; effects are percentage point changes; absolute t -statistics from two-way clustering by firm and fiscal year are in parentheses. Models include both coordinates, scale squared, log age, industry fixed effects, and fiscal-year fixed effects. Real-expansion and financial-policy models additionally include the lagged value of the corresponding object.

<i>Panel (a): Exact-horizon persistence</i>							
Horizon	N range	Scale			Balance		
		λ^G	λ^O	λ^E	τ^G	τ^O	τ^E
1 year	142,480–145,341	0.990	0.991	0.990	0.869	0.836	0.611
3 years	116,884–119,101	0.968	0.970	0.970	0.756	0.681	0.453
5 years	97,146–98,905	0.949	0.951	0.951	0.685	0.584	0.355
10 years	62,832–63,882	0.906	0.907	0.907	0.578	0.464	0.279
<i>Panel (b): Conditional future outcomes</i>							
Outcome	N	Event rate (%)	Current coordinate higher by one annual-movement IQR (pp)				
			Scale (0.187)		Balance (0.050)		
<i>Downside and terminal outcomes</i>							
Operating loss next year	142,480		14.0	-0.78 (32.7)		-1.77 (31.6)	
CRSP performance/liquidation exit	163,648		2.6	-0.17 (15.1)		-0.15 (7.8)	
<i>Coordinate dynamics</i>							
Scale increase, $\Delta\lambda^O > 0$	142,480		54.5	0.05 (1.1)		0.44 (7.9)	
Balance increase, $\Delta\tau^O > 0$	142,480		50.6	0.11 (3.5)		-0.95 (21.3)	
<i>Real expansion</i>							
Total-assets increase	142,480		48.8	1.22 (12.3)		0.88 (11.5)	
PPE increase	142,480		45.4	0.92 (13.5)		0.76 (11.2)	
Employment increase	137,680		55.7	0.54 (8.2)		0.56 (10.3)	
<i>Financial policy</i>							
Investment-rate increase	124,842		49.4	-0.06 (1.4)		0.27 (5.8)	
Net-distribution-rate increase	124,842		50.1	0.25 (7.6)		0.44 (10.6)	
Debt-ratio increase	142,480		51.5	0.12 (4.3)		-0.50 (13.6)	

or public-panel disappearance more generally.

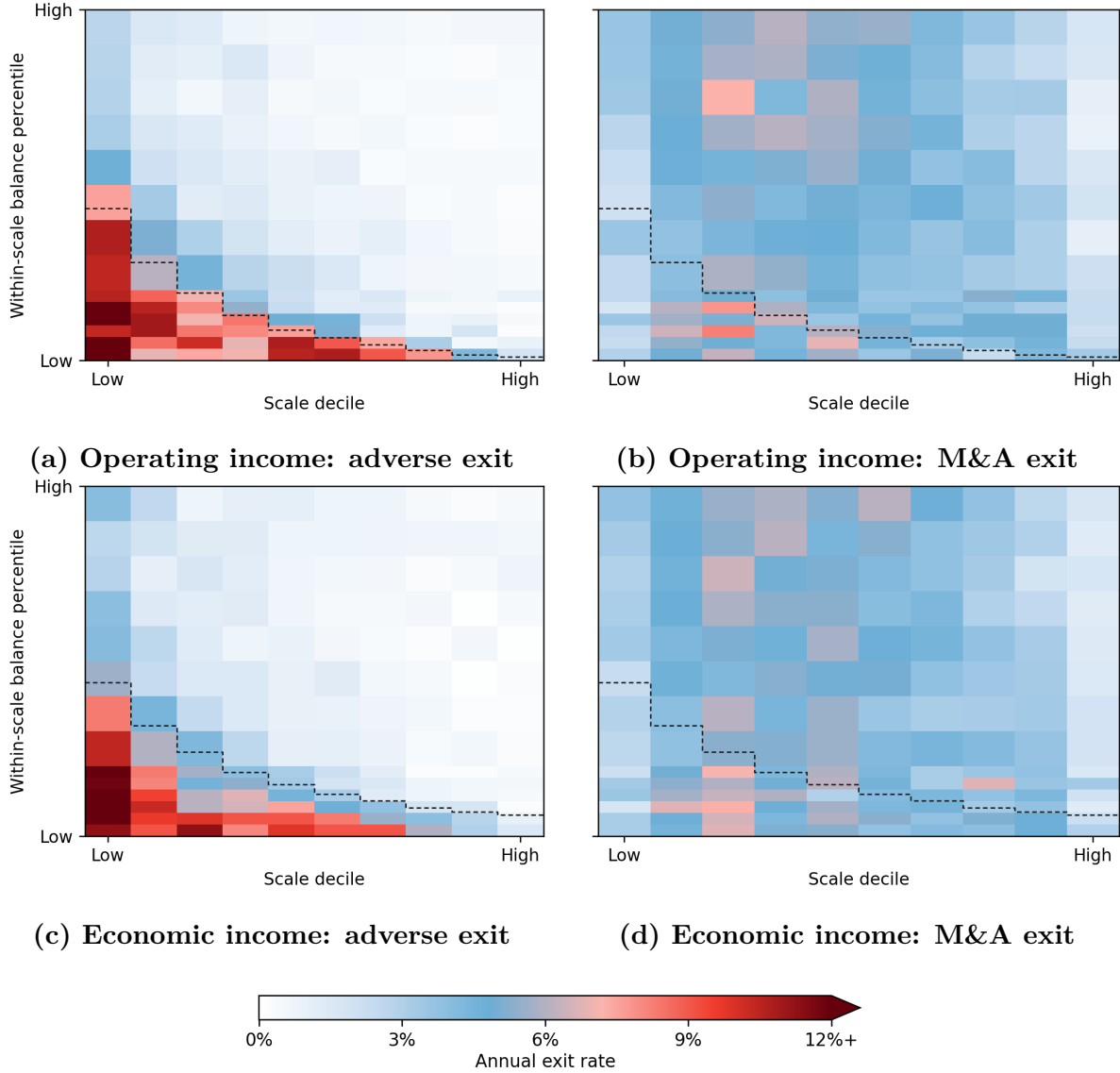


Figure 3. Classified exit over the scale–balance position. Panels plot annual CRSP exit rates. Rows compare the operating and economic-income boundaries; columns compare adverse and M&A exits. The horizontal axis orders scale deciles, and the vertical axis ranks balance within each scale decile. The lower two balance deciles are each split into terciles, preserving their percentile mass while providing finer downside resolution. The dashed step line marks break-even within each scale decile.

Alt text: Panels A and C show adverse exit rates rising sharply as balance falls below break-even within scale groups, with the highest rates among low-scale, deeply negative-balance firms. Panels B and D show merger-and-acquisition exits are comparatively diffuse and lack the same break-even gradient. The pattern appears under both income definitions.

Low balance, and especially negative balance, organizes adverse exit at both income boundaries much more sharply than it organizes M&A. And it does so for within-scale deciles. Scale shifts the

level of gross activity at any relative position, while balance marks location relative to break-even. Together, Table 2 and Figure 3 establish the empirical premise that the position space and growth space of Figure 1 are economically meaningful and organize firm outcomes.

3.3 The Gaussian input benchmark and the tails of income growth

Next, I consider the constructive benchmark in Corollary 1, which establishes the emergence of heavy-tailed income growth as a consequence of the geometry of sales and expenses, or equivalently scale and balance. I first estimate a deliberately simple dynamic law summarizing their movement, and then simulate this law with Gaussian innovations, to create simulated income and income growth draws. Both income and its positive-to-positive conventional growth are heavy-tailed in the simulation, though less so than in the data. The DLN distribution again exhibits a remarkable fit to the tails of both as observed in the data. Finally, I discuss the probable source of the gap between the tail weight of the Gaussian benchmark and the data.

The law of movement for $x_{it} = (\lambda_{it}, \tau_{it})'$ is a VAR(1) reduced-form law for observable position,

$$x_{i,t+1} - m = A(x_{it} - m) + u_{i,t+1}. \quad (15)$$

I estimate the center, transition matrix, and residual dispersion and correlation using the 1970–2004 sample, and keep 2005–2019 held out. I calculate residual dispersion by converting innovation IQRs to Gaussian standard deviations, retaining their correlation.

The fitted central law is stable, robust, and two-dimensional. For operating income,

$$\hat{A}^O = \begin{pmatrix} 0.991 & 0.018 \\ 0.005 & 0.807 \end{pmatrix}, \quad (\hat{\sigma}_\lambda, \hat{\sigma}_\tau) = (0.137, 0.041),$$

with innovation correlation 0.197. For economic income,

$$\hat{A}^E = \begin{pmatrix} 0.987 & 0.160 \\ 0.012 & 0.587 \end{pmatrix}, \quad (\hat{\sigma}_\lambda, \hat{\sigma}_\tau) = (0.141, 0.057),$$

with innovation correlation -0.015 . These estimates supply a simple, transparent input law against

which to evaluate the nonlinear consequences of the signed-income map.

I impose Gaussian innovations on the fitted laws of motion, solve the discrete Lyapunov equations to determine the stationary distribution of the dynamic process, and draw 300,000 simulated one-year income transitions. Coordinate positions and native movements are jointly-Normal by construction. Figure 4 displays the results of the benchmark experiment. The position and movement inputs in Panels A and B follow the Normal reference line; their signed-income and positive-income-growth outputs in Panels C and D have pronounced heavy tails. The simulated income growth distribution is somewhat thinner than the data but far thicker than a Normal. A marginal DLN fit closely tracks the tails of the data distributions.

Table 3 quantifies the tail-incidence across income boundaries. Panel A shows that across all boundaries: (i) both income and conventional income growth exhibit considerable heavy-tails in the data; (ii) the Gaussian simulation also exhibits strong heavy-tails for both, but less than in the data; and (iii) the DLN fit to the data matches the data’s tail-weight very closely. A regular Gaussian law for two economically interpretable inputs is therefore sufficient to produce substantial income growth amplification and heavy-tails.

Panel B then rationalizes the difference between the Gaussian VAR and the data tails. The Gaussian VAR explicitly assumes the innovations are Normal in order to establish sufficiency. But Panel B shows that the innovations $u_{i,t+1}$ (and similarly $(\Delta\lambda, \Delta\tau)$) are themselves heavy-tailed. Panel B shows their tail incidence is tracked well by their fitted DLN distributions, and unreported results show they, too, show remarkable DLN fit. Heavy-tailed empirical scale and balance innovations feed additional extreme native movements into the same nonlinear map. Geometry and native innovation heavy tails are hence distinct, cumulative sources of the heavy tails of the empirical distribution.

Scale and balance are observable, economically active coordinates with distinct dynamics over the full signed domain. Their scalar projection is amplified near break-even. Gaussian position and movement already generate heavy-tailed signed income and income growth, and the empirical innovations add native tail variation. Section 4 now asks how this distinction changes familiar finance concepts such as growth coefficients, valuation–growth relations, and equity–return–tail evidence.

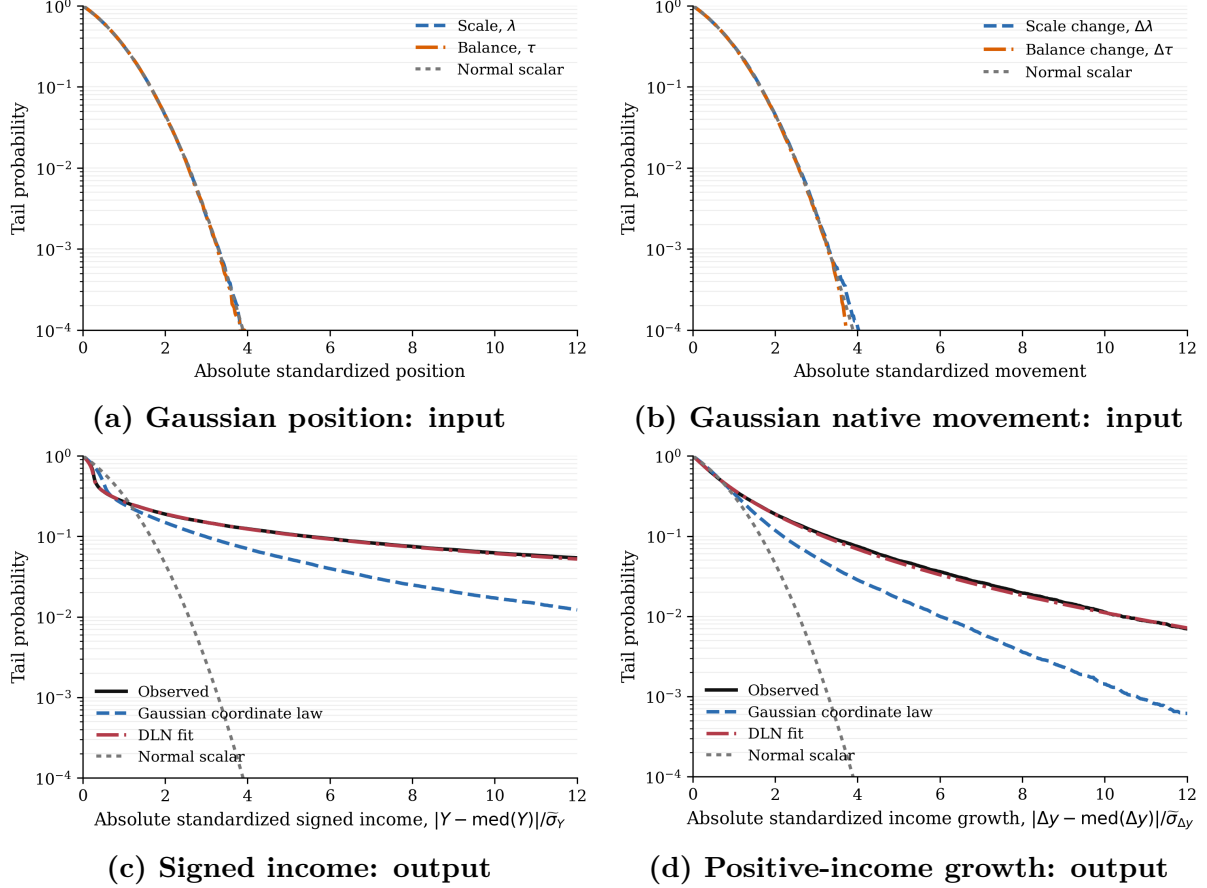


Figure 4. **Gaussian coordinates produce heavy-tailed income and income growth.** Panels A and B plot stationary operating scale and balance positions and their native changes under the fitted Gaussian coordinate law. Panel C plots signed operating income, and Panel D plots $\Delta y = \log Y_{t+1} - \log Y_t$ for transitions with $Y_t, Y_{t+1} > 0$. Blue curves are generated from the analytical invariant distribution of the Gaussian VAR estimated over 1971–2004; black curves report held-out 2005–2019 observations. Red dash-dot curves are DLN fits to the corresponding held-out outputs. Every curve is centered at its own median and divided by its own IQR-implied standard deviation, so the panels compare distributional shape rather than units. Gray curves report absolute standard-Normal tail probabilities.

Alt text: Panels A and B show simulated scale and balance positions and changes following the Normal tail benchmark. Panels C and D show signed income and positive-income growth developing much heavier tails from those Gaussian inputs. Held-out data are heavier still, while fitted difference-of-lognormals curves closely follow the observed tails.

Table 3. **Gaussian inputs and heavy-tailed outputs across income boundaries.** A tail is $|X - \text{med}(X)| > 3\tilde{\sigma}_X$, where $\tilde{\sigma}_X = \text{IQR}(X)/1.34898$, calculated separately by object, law, and income boundary; the corresponding analytical Normal rate is 0.27%. Panel A compares held-out 2005–2019 outputs with the invariant Gaussian coordinate law and DLN fits. Positive-income growth is $\Delta y = \log Y_{t+1} - \log Y_t$, conditional on $Y_t, Y_{t+1} > 0$. Panel B defines scale and balance innovations as the untrimmed one-year residuals from the robust centered coordinate VAR(1) estimated over 1971–2004 and reports their marginal DLN fits.

Object	Law	Gross	Operating	Economic
<i>Panel A: Income and income-growth outputs</i>				
Signed income	Observed	14.13%	14.92%	15.64%
	Gaussian coordinate law	7.92%	9.85%	10.62%
	DLN fit	14.05%	14.89%	16.06%
Positive-income growth	Observed	9.56%	11.25%	8.58%
	Gaussian coordinate law	1.28%	5.40%	5.30%
	DLN fit	9.31%	10.79%	7.79%
<i>Panel B: Empirical native innovations</i>				
Scale innovation, u_λ	Observed	7.87%	7.62%	7.52%
	DLN fit	7.80%	7.45%	7.31%
Balance innovation, u_τ	Observed	11.13%	13.14%	14.46%
	DLN fit	11.69%	12.95%	13.57%

4 How Complete-Domain Growth Changes Finance Evidence

The section presents three applications of the theory. Each application follows the same empirical comparison: I first reproduce the conventional conclusion on the censored positive-to-positive sample for which conventional income growth is defined. I then use scale and balance to identify the native movement observed on the same sample, to show both the impact of mechanical amplification near $\tau = 0$ and the fuller understanding of firm dynamics afforded by using the growth vector. Finally, I restore the loss-entry (p-to-n), continued-loss (n-to-n), and recovery (n-to-p) transitions for which conventional income growth is undefined, and estimate the outcomes on the uncensored sample. This sequence separates two consequences of scalar growth measurement: the nonlinear projection changes the movement attributed to a firm, and the positive-income domain changes the firms represented by the estimand.

The section applies this comparison to three familiar finance questions. First, it asks what common finance predictors forecast in the selected and complete-domain growth spaces. Second, it revisits the persistence and valuation of income growth. Third, it locates omitted income-sign transitions and extreme native movements in realized equity-return tails, showing that much of the equity return tail stems from the censored part of the income-growth sample.

4.1 Finance predictors in selected and complete-domain growth

Finance research routinely uses market value, investment, cash flow, debt, and asset composition to characterize subsequent firm income growth. The conventional income-growth outcome, however, exists only when income is positive at both endpoints. The coordinates first reveal scale and balance change on that same selected sample and then extend the outcomes to every observable income-sign transition. Panel a of Table 4 reports what the conventional positive-to-positive sample says about year-ahead income growth. Some familiar Δy coefficients are far larger than the native movements observed on the same firms. At the economic-income boundary, a one-IQR increase in V/K predicts 25.10 log points of conventional income growth but only 2.47 points of scale change and a 1.82-point half-balance change. Cash flow predicts -31.76 points of Δy , compared with -2.66 points of scale change and a -1.95 -point half-balance change. These gaps are the empirical counterpart of income growth amplification: the nonlinear balance contribution converts ordinary native movement into

an extreme measured response.

The conventional growth measure can also change the sign of the economic reading. At the economic-income boundary, net debt predicts 7.96 points of Δy and PPE predicts 9.22 points, even though they predict scale decreases of 1.10 and 0.98 points, respectively. Their positive scalar coefficients do not describe firms becoming larger in the revenue–cost system. They describe firms with shrinking scale, balance improvement, and again the effect of mechanical amplification of balance improvement near low τ .

Panel b then presents the same growth regressions over the full, uncensored sample. conventional income growth is no longer defined so the panel instead reports revenue growth. Because $\Delta r = \Delta \lambda + \Delta \tau / 2$, its complete-domain coefficient provides an exact adding-up check on the two native coefficients. Indeed, every Δr coefficient in Panel b is the exact addition of the $(\Delta \lambda, \Delta \tau)$ coefficients in the next two columns. The coefficients on expense growth will mechanically be their difference as well. The finance question is how our understanding on income growth changes when we move to the full sample and consider the two components independently.

A notable difference between Panels a and b is that cashflow loses much of its predictive power. From one of the strongest predictors of income growth it becomes one of the weakest when the entire sample is included. The opposing effects of net-debt are further amplified, making it a strong predictor of scale decline as well as balance growth. Similarly, tangibility’s impact on scale changes sign between the samples, and its impact on balance increases, making it the strongest predictor of operating balance growth. The negative impact of investment on subsequent balance growth increases nearly 7-fold between the samples for economic income, and changes sign for operating income. Current loss, which the selected positive-to-positive sample cannot contain, predicts a 4.85-point economic scale decrease. Selection can therefore preserve, attenuate, or change the sign of the native relation, depending on the finance predictor.

Figure 5 maps the complete economic-income specification from Panel B into the coefficient or growth plane. Four finance variables are the “poster-child” of each quadrant. Investment strongly increases scale, as expected, but reduces balance, that is makes the firm larger but with lower profitability. Debt has the opposite effect, reducing scale and increasing balance. Value predicts both increasing scale and increasing balance, while cashflow predicts a small decrease in both. Tangibility is nearly all-balance, and loss is nearly all-scale. The locations on the figure distinguish

Table 4. **Finance predictors in selected and complete-domain income growth.** The table reports one-year-ahead regressions. Panel A restricts each income boundary $j \in \{O, E\}$ to positive-to-positive transitions and reports conventional income growth Δy^j , scale change, and half-balance change on identical observations. The native columns do not add to Δy^j , whose balance contribution is nonlinear. Panel B retains every otherwise eligible coordinate transition and replaces Δy^j with revenue growth; within each boundary, $\hat{\beta}_r = \hat{\beta}_\lambda + \hat{\beta}_\tau/2$. All models contain the five displayed finance variables, $\log(1 + Age_t)$, current position, lagged native movement, above-reference-balance and loss indicators, FF48 effects, and fiscal-year effects. Continuous predictors use complete-domain fiscal-year medians and IQRs held fixed across panels; indicators remain in levels. Coefficients are log points, with absolute t -statistics from two-way firm and fiscal-year clustered covariance in parentheses. The loss indicator is unidentified in Panel A by construction. Incremental adjusted R^2 is the gain from adding current position and lagged native movement to the transformed-finance baseline on the identical panel-specific sample.

	Operating-income boundary			Economic-income boundary		
<i>Panel (a): Conventional positive-to-positive sample</i>						
Predictor	Δy^O	$\Delta \lambda^O$	$0.5\Delta \tau^O$	Δy^E	$\Delta \lambda^E$	$0.5\Delta \tau^E$
$\log(V_t/K_t)$	11.28 (16.2)	3.95 (16.1)	0.50 (12.2)	25.10 (17.4)	2.47 (11.5)	1.82 (12.5)
$\text{asinh}(I_t/K_{t-1})$	5.34 (11.3)	5.57 (24.1)	0.06 (2.1)	7.18 (12.7)	6.24 (27.7)	-0.21 (2.3)
$\text{asinh}(CF_t/K_{t-1})$	-8.78 (18.8)	-2.39 (16.4)	-0.33 (9.9)	-31.76 (21.6)	-2.66 (12.7)	-1.95 (10.9)
$\text{asinh}(ND_t/K_t)$	1.59 (3.1)	-0.51 (3.1)	0.14 (4.5)	7.96 (8.2)	-1.10 (5.8)	0.96 (11.6)
$\log(PPE_t/K_t)$	4.94 (12.3)	-0.12 (0.7)	0.40 (15.3)	9.22 (16.6)	-0.98 (5.0)	0.87 (10.7)
Loss indicator	—	—	—	—	—	—
Adjusted R^2	0.061	0.165	0.090	0.176	0.217	0.267
Incremental adjusted R^2	0.025	0.005	0.062	0.037	0.047	0.139
Observations	103,978	103,978	103,978	92,390	92,390	92,390
<i>Panel (b): Complete-domain coordinate sample</i>						
Predictor	Δr	$\Delta \lambda^O$	$0.5\Delta \tau^O$	Δr	$\Delta \lambda^E$	$0.5\Delta \tau^E$
$\log(V_t/K_t)$	4.76 (15.8)	4.14 (16.7)	0.62 (6.4)	4.54 (15.0)	2.81 (12.3)	1.73 (10.1)
$\text{asinh}(I_t/K_{t-1})$	4.93 (24.1)	5.36 (26.6)	-0.43 (5.3)	5.11 (26.0)	6.43 (34.1)	-1.33 (11.7)
$\text{asinh}(CF_t/K_{t-1})$	-0.89 (5.5)	-0.94 (7.2)	0.05 (0.9)	-0.81 (3.2)	-0.42 (1.7)	-0.39 (2.8)
$\text{asinh}(ND_t/K_t)$	-0.33 (1.2)	-0.97 (4.8)	0.64 (6.0)	-0.28 (1.0)	-1.99 (8.6)	1.71 (10.8)
$\log(PPE_t/K_t)$	1.21 (5.0)	0.56 (2.7)	0.65 (9.2)	1.14 (4.9)	0.06 (0.3)	1.08 (8.4)
Loss indicator	-6.63 (10.5)	-7.23 (13.9)	0.60 (2.4)	-4.62 (11.0)	-4.85 (11.3)	0.23 (0.6)
Adjusted R^2	0.108	0.136	0.121	0.104	0.192	0.252
Incremental adjusted R^2	0.012	0.008	0.083	0.008	0.025	0.115
Observations	124,842	124,842	124,842	124,842	124,842	124,842

economically different directions of firm movement without reducing them to a scalar ordering.

4.2 Persistence and valuation of income growth

4.2.1 Persistence across income-sign transitions

The conventional persistence test asks whether past income growth predicts future income growth, but it can follow only histories with positive income at each measurement date. Figure 6 first compares conventional income growth with native scale and balance change over identical selected histories, then restores every observable income-sign transition.

The figure changes the economic interpretation of income-growth mean reversion. In Panels a and b, conventional positive-to-positive income growth mean reverts. On the same observations, scale change is positively persistent while balance change reverses. The familiar mean reversion of income growth is therefore balance reversal alongside persistent scale increase, with both taking place quickly and then persisting over longer horizons. Panel c then presents the scale and balance persistence for both operating and economic income over the full uncensored sample. The two share nearly identical persistence patterns, again show positive scale persistence along with balance reversion. Panel d decomposes the operating scale persistence from panel c into its four components, showing that transition history matters: operating-scale change remains persistent after p-to-p movement but is reversing after loss entry (p-to-n).

4.2.2 Valuation and subsequent income growth

The valuation–growth literature asks whether current market multiples distinguish firms with unusually high subsequent income growth [Lakonishok et al., 1994, Chan et al., 2003, Sloan and Wang, 2025, Tengulov et al., 2025]. Chan, Karceski, and Lakonishok measure long-horizon sales and income growth mainly per share and restrict conventional income growth to a positive base. Sloan and Wang refine the recurring operating-EPS object, while Tengulov, Zechner, and Zwiebel emphasize EBITDA and sales. These tests ask whether high-valuation firms subsequently deliver the growth embedded in prices, but their income-growth conclusions necessarily describe a selected positive-income population or a different positive performance object.

The scale–balance system extends that question to the complete signed domain. I hold the

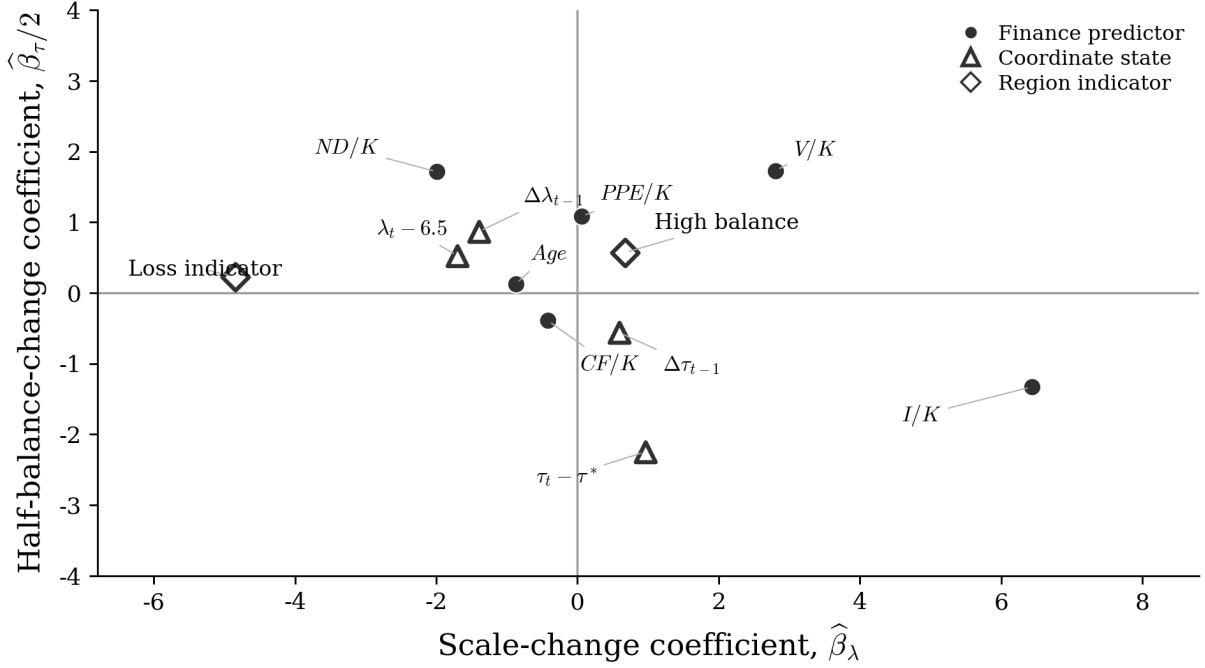


Figure 5. **Finance predictors in the complete-domain growth plane.** The figure maps every coefficient from the augmented economic-income regressions in Panel B of Table 4. The horizontal coordinate is the scale-change coefficient and the vertical coordinate is the half-balance-change coefficient, both in log points. Filled circles denote finance predictors, open triangles coordinate states, and open diamonds region indicators. Labels name the underlying economic objects; transformations, scaling, and inference are those of the regressions in Table 4.

Alt text: Predictors are positioned by scale change horizontally and half-balance change vertically. Investment lies far right and below, indicating scale increase with balance deterioration. Firm value lies upper right, net debt upper left, and cash flow slightly lower left. Tangibility primarily predicts balance improvement, while the loss indicator primarily predicts scale decline.

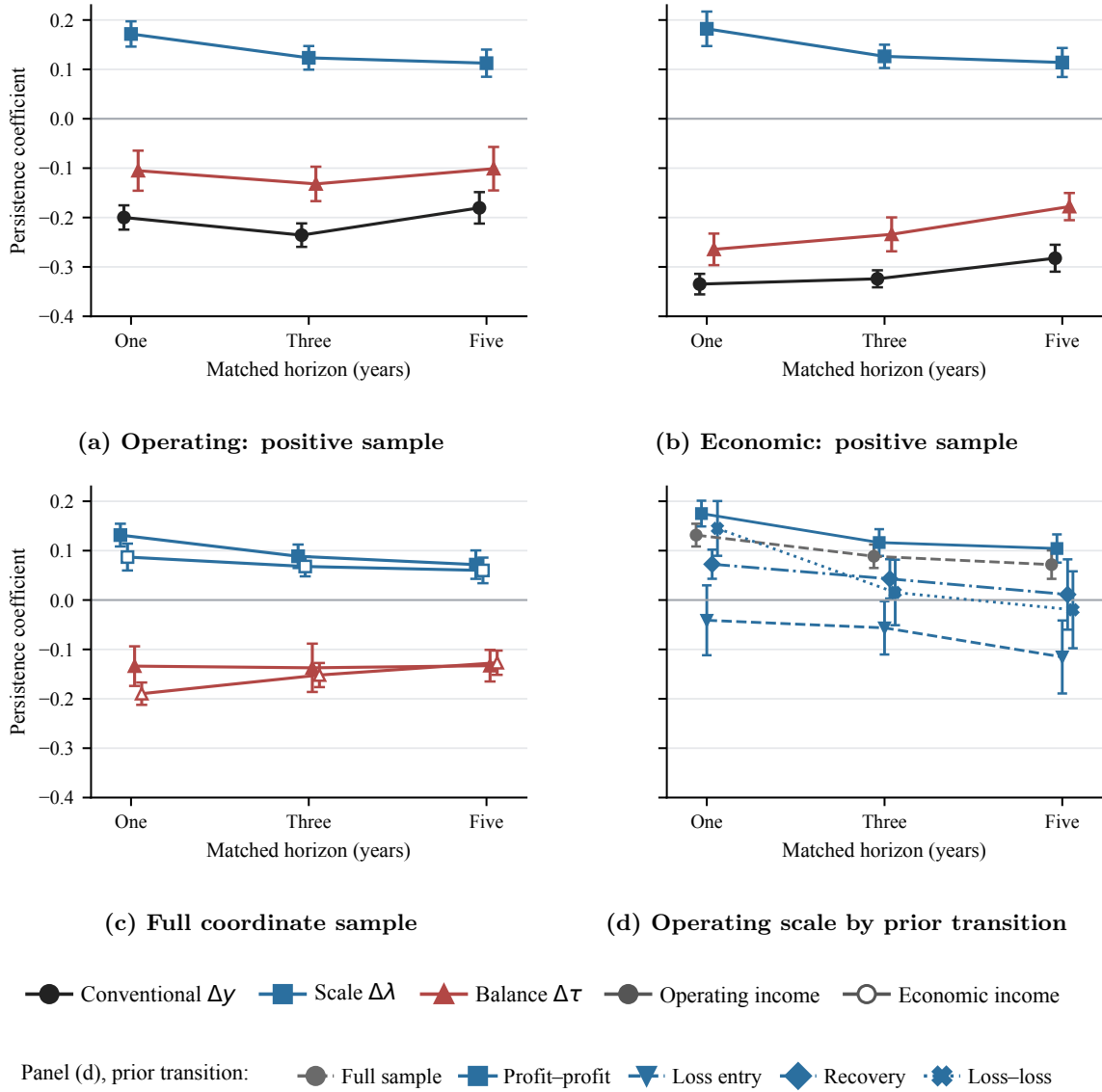


Figure 6. **Income-growth persistence and prior income-sign transitions.** At horizon h , past and future growth are annualized over the adjacent intervals from $t - h$ to t and from t to $t + h$. Panels (a) and (b) require positive income at all three measurement dates and report conventional income growth, scale change, and balance change. Panel (c) uses the full coordinate-regression sample and reports the coordinate laws for both income definitions. Panel (d) estimates operating-scale persistence separately according to the income-sign transition completed from $t - h$ to t . Intermediate income signs are unrestricted. Bars show 95 percent confidence intervals based on firm- and year-clustered standard errors.

Alt text: Panels A and B show conventional positive-income growth reversing while scale changes persist and balance changes reverse. Panel C retains the same scale-persistence and balance-reversal pattern after restoring all sign transitions. Panel D shows operating-scale persistence is positive after continued profit but negative after loss entry; recovery and continued loss exhibit weaker, fading persistence.

ex-ante valuation portfolios fixed, reproduce the conventional positive-to-positive income-growth estimand, and then retain every observed signed-income endpoint. Parallel operating-, economic-, and per-share-income results separate domain selection from the income definition and EPS normalization.

Table 5 shows that familiar valuation signals forecast economically large scale increases that conventional income growth can largely conceal. High-minus-low q firms have only 2.19 annualized log points more operating-income growth in the positive-to-positive sample. On those same observations, they have 8.09 points more scale change and a -5.90 -point balance contribution. Holding the ex-ante portfolios fixed and restoring all observed signed endpoints raises the operating-scale spread to 9.19 points.

High valuation therefore does predict substantial subsequent operating-scale increase; worsening balance makes the bottom-line growth difference appear small. Sales-to-price gives the same pattern, whereas earnings-to-price loads on both scale and balance. The same interpretation appears across operating income, economic income, and economic income per share. The balance contribution vs. balance change columns again explicitly show the impact of magnification.

4.3 Income growth and equity-return tails

Conventional income growth omits every transition into, out of, or within loss. Equity-return tails provide a direct finance test of whether those omitted histories are peripheral or instead occur when shareholder outcomes are most extreme. The earnings-returns literature establishes that accounting performance contains information reflected in equity prices [Ball and Brown, 1968, Easton and Harris, 1991]. Earnings levels and changes scaled by beginning market value remain defined for losses; they measure an earnings-return association over the signed domain. Basu uses returns as firm news to document asymmetric recognition of bad news, while Vuolteenaho separates firm-level returns into cash-flow and discount-rate news [Basu, 1997, Vuolteenaho, 2002]. The scale-balance representation adds the complete signed-income movement accompanying the realized return.

Table 6 shows that omitted income-sign transitions are economically selected. Positive-to-positive transitions constitute 81.0 percent of matched operating observations but only 33.2 percent of the lower return-tail mass; the corresponding economic-income shares are 72.6 and 20.2 percent.

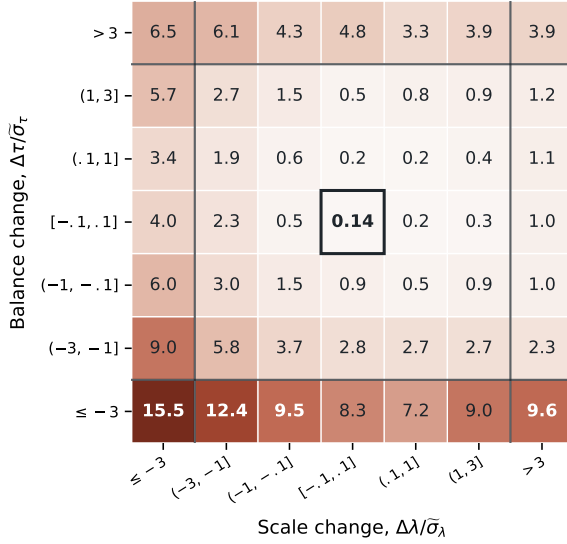
Table 5. **Valuation and the anatomy of five-year income growth.**

	Positive-to-positive endpoints				All endpoints	
	Income growth Δy	Scale change $\Delta \lambda$	Balance contribution $\Delta y - \Delta \lambda$	Balance change $\Delta \tau$	Scale change $\Delta \lambda$	Balance change $\Delta \tau$
<i>Panel (a): Operating income</i>						
H-L q	2.19 (3.94)	8.09 (16.43)	-5.90 (-9.90)	-0.82 (-14.49)	9.19 (15.59)	-0.62 (-4.55)
L-H S/P	4.12 (6.17)	9.46 (23.77)	-5.34 (-10.33)	-0.90 (-12.22)	10.58 (21.04)	-0.07 (-0.22)
L-H E/P	15.26 (15.01)	7.61 (14.65)	7.65 (11.29)	0.74 (5.27)	6.80 (14.41)	0.15 (0.86)
<i>Panel (b): Economic income</i>						
H-L q	0.81 (1.14)	7.39 (14.65)	-6.59 (-12.54)	-0.49 (-4.49)	9.23 (16.66)	-0.80 (-4.42)
L-H S/P	3.29 (4.87)	9.17 (21.57)	-5.88 (-12.91)	-0.70 (-3.60)	10.62 (21.10)	-0.16 (-0.59)
L-H E/P	21.13 (15.70)	4.70 (8.77)	16.42 (13.76)	1.96 (6.07)	4.36 (8.17)	1.60 (4.80)
<i>Panel (c): Economic income per share (economic EPS)</i>						
H-L q	1.27 (1.46)	7.86 (12.62)	-6.59 (-12.54)	-0.49 (-4.49)	9.09 (11.42)	-0.80 (-4.42)
L-H S/P	3.73 (4.34)	9.62 (14.36)	-5.88 (-12.91)	-0.70 (-3.60)	9.82 (15.38)	-0.16 (-0.59)
L-H E/P	22.79 (15.10)	6.36 (9.62)	16.42 (13.76)	1.96 (6.07)	5.86 (8.90)	1.60 (4.80)

Description. Entries are five-year annualized high-valuation-minus-low-valuation spreads in log points, multiplied by 100; Newey–West t -statistics with four lags appear in parentheses. Valuation quintiles are formed annually from unique firm-year origins. High valuation means high q , low sales-to-price, or low earnings-to-price; E/P sorts use profitable origins and the income object named by the panel. The first four columns use identical positive-to-positive observations. The balance contribution is $\Delta y - \Delta \lambda = \Delta \log[2 \sinh(\tau/2)]$, while $\Delta \tau$ is native balance change. The final two columns retain every observed signed-income endpoint using the same ex-ante portfolios. Economic EPS is economic income divided by fiscal-year-end split-adjusted shares N , so $\lambda^{E,ps} = \lambda^E - \log N$ and $\tau^{E,ps} = \tau^E$. Depending on the panel and sort, the two samples contain 21,225–29,472 and 30,578–36,131 firm-horizon observations, respectively. Origins run through 2014, so all endpoints fall by 2019.

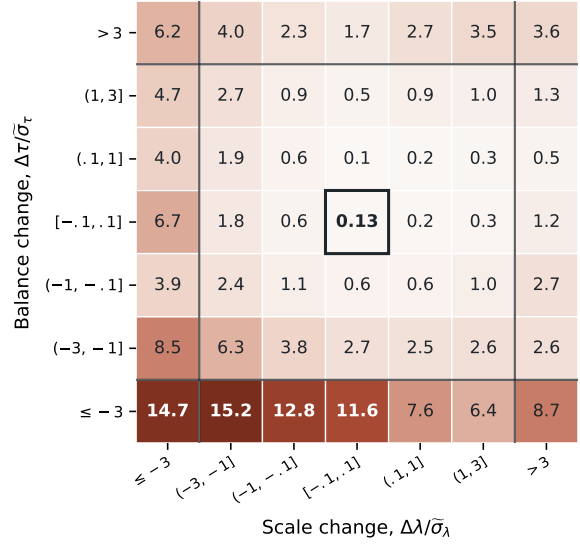
Table 6. **Income-sign transitions and equity-return tails.** Panel A partitions matched one-year transitions by beginning and ending income sign. Panel B repeats the partition for extreme scale or balance changes. Panel C compares alternative extreme definitions; its rows can overlap. The All column reports each row’s share of matched transitions; Lower tail and Upper tail report its shares of the corresponding realized return tails. Extreme Δy is defined only for positive-to-positive transitions, whereas coordinate extremes use all matched coordinate transitions. Extremes exceed three IQR-implied standard deviations.

<i>Panel (a): Income-sign transitions in the matched sample</i>								
	Operating income				Economic income			
	Share (%)				Share (%)			
	<i>N</i>	All	Lower tail	Upper tail	<i>N</i>	All	Lower tail	Upper tail
Positive $\rightarrow$ positive	107,403	81.0	33.2	64.3	94,651	72.6	20.2	57.4
Positive $\rightarrow$ negative	6,607	5.0	22.3	2.6	11,708	9.0	32.4	3.5
Negative $\rightarrow$ positive	5,580	4.2	3.7	15.5	10,543	8.1	6.3	24.6
Negative $\rightarrow$ negative	13,077	9.9	40.9	17.6	13,529	10.4	41.1	14.5
Total	132,667	100.0	100.0	100.0	130,431	100.0	100.0	100.0
<i>Panel (b): Income-sign transitions with extreme $\Delta\lambda$ or $\Delta\tau$</i>								
Positive $\rightarrow$ positive	10,621	8.0	10.9	19.2	9,812	7.5	6.1	17.1
Positive $\rightarrow$ negative	4,027	3.0	16.8	1.7	5,740	4.4	23.8	2.3
Negative $\rightarrow$ positive	3,240	2.4	2.6	11.9	5,229	4.0	4.4	15.8
Negative $\rightarrow$ negative	7,663	5.8	27.6	13.1	6,650	5.1	27.0	10.3
Total	25,551	19.3	57.9	46.0	27,431	21.0	61.3	45.4
<i>Panel (c): Alternative definitions of extreme movement</i>								
Extreme Δy	11,444	8.6	12.1	21.6	7,996	6.1	4.6	11.7
Extreme $\Delta\lambda$	10,259	7.7	21.7	19.4	10,234	7.8	22.3	20.8
Extreme $\Delta\tau$	20,702	15.6	52.2	37.6	23,481	18.0	56.6	36.6
Extreme $\Delta\lambda$ and $\Delta\tau$	5,410	4.1	16.0	11.1	6,284	4.8	17.7	12.0



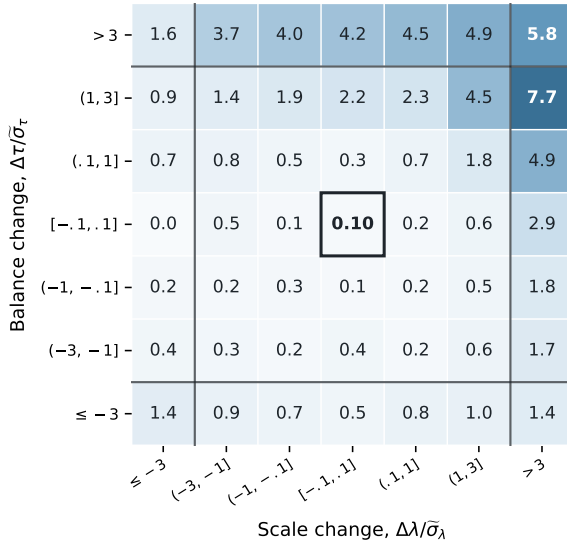
(a) Operating income: lower tail

Full sample: 2.32%; Gaussian: 0.135%



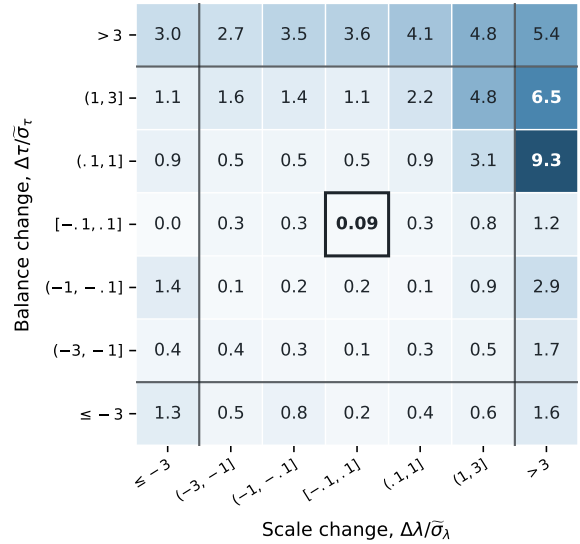
(b) Economic income: lower tail

Full sample: 2.25%; Gaussian: 0.135%



(c) Operating income: upper tail

Full sample: 1.10%; Gaussian: 0.135%



(d) Economic income: upper tail

Full sample: 1.08%; Gaussian: 0.135%

Figure 7. **Equity-return tails over the scale–balance growth plane.** Each cell gives the incidence of the indicated contemporaneous market-adjusted equity-return tail among firm-years in that region of the one-year growth plane. A return enters a tail when it is more than three within-year IQR-implied standard deviations from the annual median. Panel subtitles give the unconditional incidence and the 0.135-percent one-sided Gaussian benchmark. Axes normalize scale and balance changes by income-boundary-specific IQR-implied standard deviations. The outlined center contains observations within 0.10 units of zero on both axes; outer bins begin at three units. Colors are comparable within each row.

Alt text: Panels A and B show lower-return-tail incidence near the Gaussian benchmark at the center but rising sharply with large negative balance changes, especially alongside scale contraction. Panels C and D show upper-return tails concentrated in large positive balance changes combined with scale increases. The patterns are similar for operating and economic income.

The missing lower-tail mass is concentrated in loss entry and continued loss. At the operating-income boundary, those transitions represent 14.9 percent of observations but 63.2 percent of the lower tail; at the economic-income boundary, they represent 19.4 percent and 73.5 percent. Recovery supplies the corresponding upside result: it represents 4.2 percent of operating observations and 15.5 percent of the upper tail, with economic-income shares of 8.1 and 24.6 percent.

Extreme native movement further concentrates shareholder tails. Extreme scale or balance change accounts for 19.3 percent of operating observations and 21.0 percent of economic observations, yet roughly 60 percent of the lower tails and 45 percent of the upper tails. Figure 7 supplies the geometry. Equity tail incidence is close to the Gaussian benchmark near the center of the growth plane and rises sharply in particular directions away from the origin. Operating and economic income produce the same qualitative map. Conventional growth therefore omits the loss and recovery histories that are disproportionately represented in extreme shareholder outcomes.

Taken together, the applications deliver one conclusion through repeated comparisons. Within the positive-to-positive sample, conventional income growth can amplify native movement or have the opposite sign from scale change. This changes economic interpretation even before extending the sample. Restoring the omitted income-sign transitions can also change the native relation between finance predictors and firm movement. Conventional mean reversion resolves into persistent operating-scale change and reversing balance, while valuation portfolios reveal substantial future operating-scale increases that balance deterioration conceals. Finally, the histories excluded from conventional growth are concentrated in both downside loss and upside recovery tails. The conventional measure therefore changes both the movement attributed to the firm and the firms represented by the evidence.

5 Conclusion

Income growth is two-dimensional because income is the signed result of two positive flows. Scale records the common magnitude of revenue and cost; balance records their proportional difference and locates the firm relative to break-even. Their changes recover common and relative movement and reconstruct revenue and cost growth exactly. A single bottom-line growth rate cannot carry the same information. It reports the movement of the net outcome, not the full movement of the

revenue–cost system that generated it. The unit of growth analysis is therefore the system that creates income, not only the bottom line it reports.

Conventional income growth introduces two distinct distortions. First, it nonlinearly magnifies balance movement near break-even. The net income base becomes small while the revenue and cost base on which balance acts remains large, so ordinary component changes can produce enormous measured growth. Second, conventional log growth is unavailable whenever either endpoint is non-positive. Loss entry, continued loss, and recovery therefore disappear from the estimand. Amplification changes the movement attributed to an observed firm; censoring changes the firms and histories represented by the evidence.

The theory and data support the same representation. Scale and balance changes remain finite through zero, add over time, are invariant to the monetary unit, and recover the underlying component growth rates. No informative endpoint-only scalar satisfies the corresponding disciplines on the full signed domain, and two continuous coordinates are the minimum for a lossless representation of unrestricted two-flow movement. Empirically, firms occupy broad position and growth planes. Scale and balance have different persistence and separately organize subsequent losses, classified exit, real change, investment, distributions, and debt. They behave as economically distinct observable firm states, not merely alternative labels for income.

The representation also explains why income and income growth have such large tails. Near break-even, balance movement acts on gross activity while scale movement acts on the much smaller net. A Gaussian input benchmark consequently produces strongly heavy-tailed signed income and positive-income growth even though its coordinate positions and movements are Gaussian by construction. Empirical scale and balance innovations are heavier-tailed still and supply additional extreme movement through the same map. DLN fits closely track the observed output distributions. The lesson is economic as well as distributional: extreme income growth can arise from ordinary firm movement at a sensitive position, from extreme coordinate movement, or from both.

This distinction changes what several familiar finance results mean. Conventional income-growth mean reversion resolves into reversing balance alongside persistent scale movement. Valuation ratios predict more subsequent operating-scale increase than bottom-line growth reveals: among high-minus-low q firms, 2.19 annualized log points of conventional operating-income growth combine 8.09 points of scale increase with a -5.90 -point balance contribution, and scale increase

risers to 9.19 points when all signed endpoints are retained. The omitted histories are also economically central to shareholders. Loss entry and continued loss contain most lower equity-return-tail observations, while recovery is disproportionately common in the upper tail. Extreme scale or balance movements constitute about one-fifth of observations but roughly three-fifths of lower-tail mass. The selected scalar changes both the interpretation of measured growth and the population from which that interpretation is drawn.

Profit, loss entry, continued loss, and recovery should therefore be studied as regions and transitions of one income system. Scalar outcomes remain useful when a question supplies a particular denominator, anchor, or event definition. When direction, sign crossings, persistence, or extreme responses matter, researchers can report scale and balance movement alongside the net outcome. The same discipline applies to models: a model that produces revenue and cost also produces an observable position and transition law in this plane. Matching that plane is a direct empirical requirement before assigning the observed behavior to particular structural primitives.

This paper establishes the observable representation, its growth law, and the finance conclusions that follow from measuring the full signed domain. A companion paper asks which structural state systems can reproduce those positions and movements. Income records the bottom line; scale and balance reveal the firm moving behind it.

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A Proofs and Representation Results

A.1 Exact position and growth maps

Proof of Proposition 1. The linear map from (r, c) to $(\lambda, \tau) = ((r + c)/2, r - c)$ has nonzero determinant and inverse $r = \lambda + \tau/2$, $c = \lambda - \tau/2$. Exponentiation is a smooth bijection from $\mathbb{R}$ to $\mathbb{R}_{++}$, so the composed map is a smooth bijection from $\mathbb{R}_{++}^2$ to $\mathbb{R}^2$. Substituting the inverse gives

$$R - C = e^{\lambda+\tau/2} - e^{\lambda-\tau/2} = 2e^\lambda \sinh(\tau/2),$$

which proves equations (4)–(5). $\square$

Proof of Proposition 2. Taking first differences of the coordinate definitions gives equation (6); solving the same nonsingular linear system gives equation (7). A common change in the monetary unit adds the same constant to every r and c level, so it cancels from $\Delta\lambda$ and from $\Delta\tau$. Positivity of R and C keeps their logs and differences finite even when $R - C$ is zero or changes sign. Finally, both coordinates are endpoint differences, so sums over adjacent intervals telescope exactly. $\square$

A.2 Equal-income geometry

Derivation of the equal-income contour. Equation (5) gives $\sinh(\tau/2) = Y/(2e^\lambda)$. Applying the inverse hyperbolic sine and differentiating while holding Y fixed give

$$\tau(\lambda; Y) = 2 \operatorname{asinh}\left(\frac{Y}{2e^\lambda}\right), \quad \frac{\partial \tau}{\partial \lambda} = -2 \tanh(\tau/2). \quad (16)$$

For $Y < 0$, $\tau < 0$ and the derivative is positive: moving to higher scale along a fixed-loss contour moves balance toward zero. $\square$

A.3 Positive-domain characterization

Proof of Lemma 1. Temporal additivity implies a potential-difference representation. Fix $x_0 = 1$ and define $h(x) = G(1, x)$, so $G(x, y) = h(y) - h(x)$. Unit invariance and $h(1) = 0$ imply $h(ay) = h(a) + h(y)$ for $a, y > 0$. Writing $q(u) = h(e^u)$ gives the additive Cauchy equation $q(u + v) = q(u) + q(v)$. Continuity yields $q(u) = \kappa u$, hence $h(x) = \kappa \log x$ and $G(x, y) = \kappa \log(y/x)$.

The normalization $G(1, e) = 1$ sets $\kappa = 1$, yielding $G(x, y) = \log y - \log x$. $\square$

A.4 Signed-domain triviality

Proof of Theorem 1. Define $h(x) = G(0, x)$. Temporal additivity gives

$$G(x, y) = h(y) - h(x). \quad (17)$$

For every $a > 0$, unit invariance applied to $(0, y)$ yields $h(ay) = h(y)$. Taking $y = 1$ makes h constant on the positive ray; taking $y = -1$ makes it constant on the negative ray. Continuity at zero equates both constants with $h(0)$. Thus h is constant on $\mathbb{R}$, and equation (17) gives $G \equiv 0$, which violates the normalization $G(1, e) = 1$ required of an informative log-point growth measure. $\square$

The proof requires neither directionality nor differentiability. Full continuity of G is unnecessary: it is enough that the two ray constants both converge to $h(0)$. Equality of the two one-sided limits alone is not enough. Without regularity at zero, additivity and unit invariance permit separate constants on the two rays and at zero; such an index records sign regimes but no within-regime magnitude.

A.5 External anchors

For any positive external scale a and continuous potential F , the class

$$G_a(x, y) = F(y/a) - F(x/a)$$

is temporally additive. It is unit invariant if the anchor co-scales with the income unit and directional if F is increasing. The unanchored IHS change,

$$G_{\text{IHS}}(x, y) = \text{asinh}(y) - \text{asinh}(x), \quad (18)$$

corresponds to fixing $a = 1$ in the reported units and therefore is not intrinsically unit invariant. Income/assets and price-scaled changes instead supply an economic anchor. Their validity depends

on the economic content of a . The theorem therefore permits application-specific scalar growth measures while excluding an intrinsic one based only on signed-income endpoints.

A.6 Symmetric additive basis and minimal dimension

Proof of Proposition 3. Continuity and path additivity imply the continuous multivariate Cauchy equation, so $\Gamma(u) = Au$ for a 2×2 matrix A . Losslessness makes A nonsingular. Exchange symmetry makes the first row symmetric and the second antisymmetric. Requiring common growth (g, g) to be recorded as first-coordinate growth g fixes the first row at $(1/2, 1/2)$. The second row is $(b, -b)$ with $b \neq 0$. This gives equation (9); $b = 1$ is the balance-change normalization. $\square$

Proof of Proposition 4. Losslessness would require a continuous injection $s : U \rightarrow \mathbb{R}$ on an open $U \subset \mathbb{R}^2$. The set U contains a circle. Restricting s to that circle produces a homeomorphism from the circle to a compact connected subset of $\mathbb{R}$, hence a closed interval. Removing an interior point disconnects the interval, whereas removing one point from a circle leaves it connected. This contradiction rules out a continuous lossless scalar. $\square$

Differentiating equation (5) yields

$$dY = 2e^\lambda \sinh(\tau/2)d\lambda + e^\lambda \cosh(\tau/2)d\tau = (R - C)d\lambda + \frac{R + C}{2}d\tau = Yd\lambda + \frac{R + C}{2}d\tau,$$

which proves equation (12). Telescoping sums prove exact temporal aggregation of both coordinate movements.

Substituting $(\lambda_{t+1}, \tau_{t+1}) = (\lambda_t + a, \tau_t + b)$ into equation (10) and subtracting initial income gives

$$\Delta Y = 2e^{\lambda_t} \left[e^a \sinh\left(\frac{\tau_t + b}{2}\right) - \sinh\left(\frac{\tau_t}{2}\right) \right]. \quad (19)$$

For nonzero same-sign endpoints, dividing the absolute values of the two income amounts gives

$$\Delta \log |Y| = a + \log \left| \frac{\sinh((\tau_t + b)/2)}{\sinh(\tau_t/2)} \right|. \quad (20)$$

Proof of Proposition 5. For fixed nonzero b , $\sinh((\tau + b)/2) \rightarrow \sinh(b/2) \neq 0$, while $\sinh(\tau/2) \rightarrow 0$. Along a sequence satisfying $\tau(\tau + b) > 0$, both endpoints have the same sign, and equation (20)

therefore applies. Its logarithmic ratio diverges in absolute value while a remains finite. Likewise,

$$\frac{Y_{t+1} - Y_t}{|Y_t|} = e^a \frac{\sinh((\tau + b)/2)}{|\sinh(\tau/2)|} - \text{sign}(\tau),$$

whose first term diverges. □

Proof of Corollary 1. Because $\rho(A) < 1$, the VAR has the unique stationary Gaussian distribution with mean μ and covariance $\Omega = \sum_{k=0}^{\infty} A^k \Sigma (A')^k$, which is equivalently the solution to $\Omega = A\Omega A' + \Sigma$. Since $\Delta x_{t+1} = (A - I)(x_t - \mu) + \varepsilon_{t+1}$, the position-movement vector is an affine transformation of jointly Gaussian variables and is therefore jointly Gaussian. Each of $e^{\lambda_t + \tau_t/2}$ and $e^{\lambda_t - \tau_t/2}$ is lognormal, with dependence inherited from the joint Gaussian position, and their difference is signed income. Applying the same argument jointly at t and $t + 1$ makes ΔY_{t+1} the stated signed sum of four jointly lognormal terms. Neither construction is restricted to be Gaussian, and any scalar income-growth response is a nonlinear function of the same Gaussian inputs. □

B Measurement and Sample Construction

B.1 Accounting map

Table B.1 maps the paper’s empirical concepts to their source-data constructions. It covers the three income boundaries and their coordinates, the firm quantities used in the applications, the valuation and equity-tail objects introduced in the later exhibits, and the operating-loss events.

Table B.1. **Construction map for the paper’s empirical objects.** The table maps the concepts used in the paper to their source-data construction. Compustat and CRSP mnemonics appear in uppercase. Income boundary j has signed income $Y^j = R - C^j$, scale λ^j , and balance τ^j .

Paper concept	Construction	Source items	Eligibility and use
<i>Panel (a): Income boundaries and native coordinates</i>			
Revenue, R_t	$R_t = \text{SALE}_t$	SALE	Reported and strictly positive; common top line for every income boundary.
Gross income, Y_t^G	$C_t^G = \text{COGS}_t, \quad Y_t^G = R_t - C_t^G$	COGS	$C_t^G > 0$ is required to form the gross position.
Operating income, Y_t^O	$C_t^O = \text{COGS}_t + \text{XSGA}_t, \quad Y_t^O = R_t - C_t^O$	COGS, XSGA	$C_t^O > 0$ is required to form operating position.
Economic income, Y_t^E	$\Delta \text{AT}_t + \text{DVT}_t + \text{PRSTKC}_t - \text{SSTK}_t + \text{XINT}_t - \Delta \text{LT}_t; \quad C_t^E = R_t - Y_t^E$	AT, DP, DVT, PRSTKC, SSTK, XINT, LT	Equals investment plus net distributions during fiscal year t . Coordinates require $C_t^E > 0$ and an exact preceding fiscal year.
Position and movement	$\lambda_t^j = \{\log R_t + \log C_t^j\}/2, \quad \tau_t^j = \log(R_t/C_t^j);$ changes are exact one-year differences	Derived from R_t, C_t^j	$j \in \{G, O, E\}$. Income is negative exactly when $\tau_t^j < 0$; $(\Delta \lambda^j, \Delta \tau^j)$ is the native growth object.
<i>Panel (b): Firm quantities used in the applications</i>			
Capital, K_t	$K_t = \text{AT}_t$	AT	Capital denominator used in the applications.
Firm value, V_t	$V_t = \text{CSHO}_t \times \text{PRCC_F}_t + \text{LT}_t$	CSHO, PRCC_F, LT	Market equity plus debt; valuation is V_t/K_t .
Investment, I_t	$I_t = \text{AT}_t - \text{AT}_{t-1} + \text{DP}_t$	AT, DP	Flow during fiscal year t ; investment intensity is I_t/K_{t-1} .
Net distributions, DI_t	$\text{DVT}_t + \text{PRSTKC}_t - \text{SSTK}_t + \text{XINT}_t + \text{LT}_{t-1} - \text{LT}_t$	DVT, PRSTKC, SSTK, XINT, LT	Signed distributions to equity and debt claimants during year t ; scaled by K_{t-1} when used as a rate.
Cash flow, CF_t	$CF_t = I_t + DI_t = Y_t^E$	Economic-income items above	The finance-regression label for the same constructed flow as economic income; cash-flow intensity is CF_t/K_{t-1} .
Debt and net debt	$DB_t = \text{LT}_t; \quad ND_t = \text{LT}_t - \text{CHE}_t$	LT, CHE	Debt-ratio outcomes use DB_t/K_t ; specifications labeled net debt use ND_t/K_t .
Net PPE	$PPE_t = \text{PPENT}_t$	PPENT	Reported as a level, log change, or PPE_t/K_t .
Employment	$EMP_t = 1,000 \times \text{EMP}_t^{\text{Compustat}}$	EMP	Compustat reports thousands of employees; growth and levels require positive reported employment.
<i>Panel (c): Valuation, per-share, and tail objects</i>			
Market equity, EQ_t	$EQ_t = \text{PRCC_F}_t \times \text{CSHO}_t$	PRCC_F, CSHO	Same-date fiscal-year-end market equity; required to be positive for price-based valuation ratios.
Valuation ratios and portfolios	$q_t = \text{Q_TOT}_t, \quad S/P = R_t/EQ_t, \quad E/P = Y_t^j/EQ_t$	Q_TOT; SALE, PRCC_F, CSHO; Y^j	Q_TOT is the Peters–Taylor total- q . Annual quintiles are formed at the origin; high valuation means high q , low S/P , or low E/P . E/P sorts require positive origin income.
Economic income per share	$N_t = \text{CSHO}_t \times \text{ADJEX_F}_t, \quad Y_t^{E,ps} = Y_t^E/N_t, \quad \lambda_t^{E,ps} = \lambda_t^E - \log N_t$	CSHO, ADJEX_F; economic-income items	Requires positive split-adjusted shares at each endpoint; $\tau_t^{E,ps} = \tau_t^E$.

continued on next page

Table B.1. **Construction map for the paper’s empirical objects** (continued).

Paper concept	Construction	Source items	Eligibility and use
Fiscal-period equity return	$\sum_{m \in (t, t+1]} \{\log(1 + \text{RET}_{im}^{adj}) - \log[1 + \text{RF}_m + \text{MKT} - \text{RF}_m]\}$	CRSP monthly return and delisting return; FF market and risk-free factors	Months strictly after the origin statement date through the destination statement date. NYSE, AMEX, or Nasdaq common shares; at least ten observed months unless a delisting occurs.
Distributional tail diagnostic	$ X - \text{med}(X) > 3\tilde{\sigma}_X$, $\tilde{\sigma}_X = \text{IQR}(X)/1.34898$	Derived within the object, law, and income boundary	Used for the signed-income, income-growth, and innovation tail rates; the two-sided Normal benchmark is 0.27 percent.
Extreme movement and return tails	$ \Delta x > 3\tilde{\sigma}_{\Delta x}$ for $x \in \{y, \lambda, \tau\}$; return cutoffs are the destination-year median plus or minus $3\tilde{\sigma}_{ret}$	Native one-year changes and the fiscal-period return above	Δy is defined only for positive-to-positive income; coordinate extremes use every observed signed-income transition. Return cutoffs are re-estimated within destination calendar year.
Adjacent horizon growth	$g_{x,h}^- = (x_t - x_{t-h})/h$, $g_{x,h}^+ = (x_{t+h} - x_t)/h$	Exact fiscal-year endpoints	Persistence comparisons use adjacent backward and forward intervals. Conventional income growth requires positive income at all three measurement dates; coordinate growth retains every observed sign.
<i>Panel (d): Records, events, and route measurement</i>			
Firm-year record	Last dated statement for each GVKEY–FYEAR	GVKEY, FYEAR, DATADATE, FYR	Changes and horizons use exact fiscal years; no position is interpolated.
Classified public-panel exit	Performance/liquidation and M&A are separate terminal events	CRSP DLSTCD; Compustat DLRSN fallback	CRSP code hundreds digits 4–5 identify performance/liquidation and digit 2 identifies M&A; unclassified disappearance remains a separate category.
Dollar units and perimeter	2019-economy scaling; acquisition screen AQC/AT	BEA GDPA, GPDCA; Compustat AQC, AT	Static dollar positions use nominal-GDP scaling; route changes use constant 2019 dollars. The controlled origin excludes AQC/AT above 20%.

Valuation portfolios are fixed at the origin before future income endpoints are observed. The positive-to-positive and all-signed columns therefore compare domains without re-sorting firms. Contemporaneous equity returns span the same statement-date interval as the accounting transition, include delisting returns, and subtract the compounded market return in log units. Return-tail cutoffs are formed separately by destination calendar year; income and coordinate tail cutoffs are formed within the relevant income boundary.