

A Second Operating State in q-Theory:

Revenue, Cost, and Corporate Losses

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Abstract

Dynamic firm models commonly combine capital with one firm-specific operating factor. Conditional on aggregate conditions, an ordered factor places revenue-cost balance on a smooth surface of capital and scale intensity. In 142,480 firm-years, the best outcome-free surface leaves 82 percent of balance dispersion. This missing margin persists and organizes investment, value, losses, and adverse CRSP exit. In a common q-theory environment, fixed-balance and deterministic-manifold technologies generate virtually no losses, while allowing balance to move independently spans the observed region and reproduces much of the observed signed-income variation. Persistent balance is a second operating state in q-theory.

JEL classifications: D21, G31, G33, L25

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1 Introduction

Dynamic models of firm value and investment require parsimonious state vectors. A standard choice combines installed capital with one firm-specific productivity or operating-profit process, conditional on common aggregate conditions. This architecture makes investment and valuation tractable and organizes much of q-theory (Abel and Eberly, 1996; Hayashi, 1982). Signed income is the difference between positive revenue and positive cost. Firms with the same capital and income can operate at different scales and distances from break-even. I show that the standard architecture suppresses an important operating margin: firms occupy a persistently thick region around the best ordered-factor surface, and their position within that region organizes investment, value, losses, and liquidation. Allowing balance to move independently supplies a simple q-theory extension that carries this margin.

I describe log revenue and log cost using scale and revenue-cost balance:

$$\lambda = \frac{\log R + \log C}{2}, \quad \tau = \log R - \log C.$$

The inverse map gives $\Pi = R - C = 2e^\lambda \sinh(\tau/2)$. Scale raises revenue and cost together; balance measures their proportional wedge. Negative balance is a loss, zero is break-even, and positive balance is a profit. Near break-even, balance moves signed income on a large operating base. This loading is the operating-leverage mechanism carried by the balance coordinate.

The main theoretical result turns this model compression into an observable restriction. Let one smooth firm-specific operating factor generate revenue and cost conditional on log capital k and common aggregate conditions. If scale intensity $x = \lambda - k$ orders that state, then every firm generated by the model satisfies

$$\tau = h(k, \lambda - k)$$

for a smooth function h . The restriction permits nonlinear technologies and flexible interactions between capital and the operating state. Its empirical content is simple: the model leaves no conditional balance width at a given (k, x) .

This restriction isolates the operating side of several prominent models whose broader mechanisms remain valuable. [Hennessy and Whited \(2007\)](#) combine capital with a positive operating-profit factor in a rich financing environment. [Gala, Gomes, and Liu \(2020\)](#) show that investment can be represented flexibly using firm states without q , while the production block determines which operating information those states carry. [Hopenhayn \(1992\)](#) and [Luttmer \(2007\)](#) add operating costs, entry, selection, and endogenous exit to scalar productivity environments. The common implication is narrower: conditional on common aggregate conditions, one ordered firm-specific operating factor restricts revenue and cost to a one-dimensional path once capital and scale are held fixed.

[Kogan, Li, and Zhang \(2023\)](#) come closest to the operating side studied here. They model revenue and variable input cost jointly and show that cost cyclicalities hedges aggregate profitability risk, making revenue-cost composition consequential for cash-flow risk and expected returns. Conditional on aggregate conditions and capital, however, one firm-specific profitability state jointly determines both flows, so firms trace a one-dimensional revenue-cost path. I ask whether this production-side compression describes the cross section. The empirical analysis includes their corresponding revenue-minus-COGS boundary.

The central stylized fact is substantial conditional balance width. I give the ordered-factor restriction its best flexible surface using balance alone and cross-fit it by firm. In 142,480 firm-years for 13,904 nonfinancial public firms from 1971 through 2019, the residual retains 82.3 percent of the unconditional interquartile range of balance. Within fine local state cells, 74.4 percent of covered firm-years lie in cells containing both profits and losses. Firms therefore occupy a thick region around the best ordered-factor surface.

This missing margin is economically active. Its one-year persistence coefficient conditional on capital and scale intensity is 0.788, and subtracting each firm's time-series median leaves

persistence of 0.510. Within annual (k, x) cells, moving from the low to high balance-residual tercile raises next-year investment by 0.122 of beginning capital and closing value relative to closing capital by 0.413. Next-year loss falls by 25.2 percentage points and adverse CRSP exit by 3.5 points. Local width, within-firm persistence, and these policy and payoff gradients make the balance residual behave as an operating state rather than unused accounting variation.

Revenue and cost continue to distinguish firms after capital and current income are held nearly fixed. Among loss firms, high-minus-low scale-intensity investment differences range from 0.088 to 0.136 of beginning capital; firms with larger revenue and cost are also 6.7 to 10.7 percentage points less likely to report another loss next year. Among profitable firms, larger-scale, thin-margin operations carry lower current and next-year value relative to capital. Revenue-cost composition therefore distinguishes policy, valuation, and the path through break-even even on capital-income contours. The marginal-return decomposition in the model shows how scale can matter at fixed capital and income.

The calibrated models show the constructive repair. Their common q-theory environment compares three operating technologies. Constant Balance fixes balance at one value, Operating Commitment traces a deterministic manifold, and Scale and Balance spans a region. The Scale and Balance model reproduces much of the observed balance width, income-to-capital dispersion, loss rates, transformed-income-change tails, and the observed directions of the investment, value, and downside comparisons under Gaussian innovations. Freezing balance within the same solved economy eliminates losses, severe losses, and transformed-income-change tails and reduces income-to-capital IQR from 0.1160 to 0.0379 while leaving median income-to-capital nearly unchanged. Direct state-law estimates determine the dynamics of the added coordinate. At the Scale and Balance loading, the innovation covariance is full rank: its smaller eigenvalue is 14.2 percent of its larger one and innovation correlation is only 0.014. Balance therefore carries distinct reduced-form movement within the fitted state law.

I make three cumulative contributions. First, I derive a new observable model restriction

and document a stylized fact. One ordered firm-specific operating factor pins revenue-cost balance to a surface conditional on aggregate conditions, capital, and scale intensity. Firms instead retain substantial conditional balance width. This provides a direct way to evaluate whether familiar operating state vectors carry the revenue-cost variation relevant for investment, value, losses, and liquidation.

Second, I document that the balance residual is persistent and relevant for investment, value, losses, and adverse CRSP exit. This extends evidence that revenue-cost composition contains information beyond current income (Fairfield and Yohn, 2001; Frank and Gao, 2025; Soliman, 2008) and connects it to evidence on the persistence, valuation, and policy consequences of corporate losses (Denis and McKeon, 2021; Hayn, 1995; Joos and Plesko, 2005).

Third, I carry scale and balance into q-theory and connect them to operating hedge, operating leverage, capital commitments, and cost structure. Kogan et al. (2023) show that endogenous variable costs make revenue-cost composition central to cash-flow risk. Operating-leverage and real-option models show how commitments generate losses and nonlinear risk, value, and exit incentives (Carlson, Fisher, and Giammarino, 2004; Hackbarth and Johnson, 2015; Novy-Marx, 2011), while accounting evidence emphasizes asymmetric cost adjustment (Anderson, Banker, and Janakiraman, 2003). The Operating Commitment benchmark translates the commitment mechanism into observable revenue-cost space, where one-factor operating variation traces a deterministic balance manifold. The Scale and Balance model adds persistent firm-level balance movement, spans the region observed in the data, and generates signed outcomes together with the documented policy and downside ordering.

2 Operating states and the firm problem

Income is revenue minus cost. The economic question is whether installed capital along with one firm-specific stochastic operating factor can span both positive flows. I first derive

the conditional balance margin that an operating factor ordered by scale intensity suppresses. I then compare three technologies that restrict balance to a single value, a deterministic line/manifold, and a full region, inside the same dynamic investment environment.

2.1 Revenue, cost, and the one-factor restriction

Let $R_t > 0$ denote revenue, $C_t > 0$ economically relevant cost, and $K_t > 0$ installed capital, with lower case letters denoting natural logs. Operating scale and revenue-cost balance are

$$\lambda_t = \frac{r_t + c_t}{2}, \quad \tau_t = r_t - c_t. \quad (1)$$

The rotation is invertible:

$$R_t = \exp(\lambda_t + \tau_t/2), \quad C_t = \exp(\lambda_t - \tau_t/2). \quad (2)$$

Signed income, the model's economic operating payoff, is therefore

$$\Pi_t = R_t - C_t = 2 \exp(\lambda_t) \sinh(\tau_t/2). \quad (3)$$

Scale raises both revenue and cost. Balance measures their proportional wedge: $\tau < 0$ is a loss, $\tau = 0$ is break-even, and $\tau > 0$ is a profit. Near break-even, balance movement acts on a large operating base relative to income. For example, a firm with revenue of 100 and cost of 90 earns 10: a 10% increase in sales and costs will increase income by 10% as well, from 10 to 11. But a 10% increase in sales and a 10% decrease in costs will increase income by 190%. Balance thus carries an operating-leverage mechanism with direct consequences for signed income.

Suppose one smooth firm-specific operating factor z generates revenue and cost after

flexible inputs have been chosen:

$$\lambda = \Lambda(k, z), \quad \tau = T(k, z).$$

Common aggregate conditions are treated as conditioned-on inputs, so z describes the cross-sectional operating factor. Define scale intensity as

$$x \equiv \lambda - k. \tag{4}$$

When x orders the operating factor at each supported capital level, the model imposes an observable restricted balance surface. Intuitively, this condition means that a higher z , conditional on k , will lead to more income, which holds for nearly all dynamic firm models discussed here.

Proposition 1 (ordered one-factor balance surface). Let $\mathcal{S}$ be the connected support of (k, z) used by the empirical test, with each conditional section $\mathcal{Z}(k) = \{z : (k, z) \in \mathcal{S}\}$ a connected interval. Suppose Λ and T are continuously differentiable on a neighborhood of $\mathcal{S}$ and, for every supported k , the map

$$z \longmapsto \Lambda(k, z) - k$$

is one-to-one on $\mathcal{Z}(k)$ with nonzero derivative throughout that interval. Then there is a unique continuously differentiable function h on the image of the supported states such that every firm generated by the model satisfies

$$\tau = h(k, x). \tag{5}$$

Proposition 1 makes the one-factor restriction observable while permitting nonlinear production and arbitrary smooth interactions between capital and the operating factor.

Curvature changes the shape of the surface; one firm-specific operating factor ordered by scale intensity still suppresses all balance width at a given (k, x) . The empirical null is the flexible surface h . Appendix A gives the proof and discusses the support conditions.

Prominent models reach this restriction through different economic mechanisms. [Hopenhayn \(1992\)](#) and [Luttmer \(2007\)](#) combine scalar productivity with operating costs, entry, selection, and endogenous exit. These mechanisms can produce losses and exit, while the scalar operating factor still confines the joint revenue-cost position to a one-dimensional set once scale is held fixed. [Hennessy and Whited \(2007\)](#) use one positive scalar operating-profit factor within a rich financing environment. [Gala et al. \(2020\)](#) characterize investment policy using flexible functions of firm states rather than q ; their production function maintains the one scalar operating-profit factor and hence is subject to the restriction as well. The surface restriction asks whether the operating information carried by the production block spans the observable revenue-cost system. Entry, financing, selection, and flexible policy rules can enrich the firm problem around a production block driven by one firm-specific operating factor, but cannot modify the underlying operating structure.

2.2 Three operating technologies

Three technologies make the restriction concrete. Constant Balance restricts balance to a single value. Operating Commitment traces a deterministic manifold. Scale and Balance lets stochastic scale and balance span a region. Each technology describes economic income after flexible inputs have been chosen.

2.2.1 Constant Balance

Constant Balance compresses operating conditions into one positive factor:

$$\Pi_t^{CB} = Z_t K_t^\theta, \quad Z_t > 0, \quad 0 < \theta < 1. \quad (6)$$

This increasing, concave specification is standard in quantitative corporate finance ([Hennessy and Whited, 2007](#)). A revenue-cost representation writes $R_t = S_t K_t^\theta$ and $C_t = \gamma S_t K_t^\theta$, with $0 < \gamma < 1$. Then

$$\Pi_t = R_t - C_t = (1 - \gamma) S_t K_t^\theta = Z_t K_t^\theta, \quad Z_t \equiv (1 - \gamma) S_t. \quad (7)$$

Revenue and cost share their capital loading and stochastic factor. Balance is fixed at the single value $\tau_t = -\log \gamma$, and income Π_t is always positive.

2.2.2 Operating Commitment

Operating Commitment adds a recurring commitment proportional to installed capital (sometimes described as “fixed cost”), as in dynamic capacity and real-option models ([Carlson et al., 2004](#); [Hackbarth and Johnson, 2015](#); [Novy-Marx, 2011](#)):

$$\Pi_t^{OC} = Z_t K_t^\theta - f K_t, \quad f \geq 0. \quad (8)$$

The commitment permits losses when $Z_t K_t^{\theta-1} < f$ and generates the nonlinear risk, value, and exit incentives emphasized in this literature. Operating variation still comes from one stochastic factor, so firms move along a restricted balance manifold rather than across the entire region.

At the observed revenue-cost boundary, let $R_t = S_t K_t^\theta$, absorb the constant variable-cost share into $Z_t = (1 - \gamma) S_t$, and write

$$C_t = \gamma R_t + f K_t. \quad (9)$$

With $x_t = \lambda_t - \log K_t$, the cost equation implies

$$e^{x_t - \tau_t/2} = \gamma e^{x_t + \tau_t/2} + f \quad \Rightarrow \quad \tau_t = h_f(x_t), \quad (10)$$

where the positive root gives

$$h_f(x) = 2 \log \left[\frac{\sqrt{f^2 + 4\gamma e^{2x}} - f}{2\gamma e^x} \right].$$

The capital commitment bends balance toward and through break-even as it grows relative to revenue. Operating Commitment income can cross zero, but every equilibrium observation remains on this one-factor manifold.

2.2.3 Scale and Balance

The Scale and Balance model allows both positive flows to vary:

$$R_t = S_t K_t^{\theta_s}, \quad C_t = X_t K_t^{\theta_x}, \quad \Pi_t = S_t K_t^{\theta_s} - X_t K_t^{\theta_x}, \quad (11)$$

where $S_t > 0$ and $X_t > 0$ are reduced-form revenue and cost shifters. [Kogan et al. \(2023\)](#) model revenue and variable cost jointly and show that the two flows respond differently to aggregate and firm-specific profitability. Conditional on aggregate conditions and capital, however, one firm-specific operating state determines the cross-sectional position in their model, similar to the other models discussed above. The Scale and Balance model adds independent firm-level movement in the revenue-cost balance. Their gross-income boundary corresponds closely to the revenue-minus-COGS layer in the cross-boundary evidence below. Revenue-based operating shifters generally combine technical efficiency, demand, and prices ([Foster, Haltiwanger, and Syverson, 2008](#)); firm heterogeneity can also reflect costs, product appeal, markups, and product scope ([Hottman, Redding, and Weinstein, 2016](#)). The two shifters provide reduced-form operating coordinates for these different sources of firm heterogeneity.

The marginal operating return is

$$\Pi_{K,t} = \theta_s S_t K_t^{\theta_s-1} - \theta_x X_t K_t^{\theta_x-1} = \frac{\theta_s R_t - \theta_x C_t}{K_t}. \quad (12)$$

Let $\theta_\lambda = (\theta_s + \theta_x)/2$ and $\theta_\tau = \theta_s - \theta_x$. Then

$$\Pi_{K,t} = \theta_\lambda \frac{\Pi_t}{K_t} + \frac{\theta_\tau}{2} \frac{R_t + C_t}{K_t}. \quad (13)$$

Equal capital and current income can therefore carry different marginal operating returns whenever $\theta_\tau \neq 0$: at a fixed Π_t/K_t , a larger operating base $(R_t + C_t)/K_t$ changes the return to installed capital. This is the model's static composition channel. Its dynamic channel remains active even when θ_τ is small. Persistent balance changes the distribution of future signed payoffs and the firm's distance from break-even, thereby changing value, investment, and liquidation once the technology is embedded in the firm problem below.

Rotating the two shifters gives the Scale and Balance capital-adjusted state mapping

$$\lambda_t = \theta_\lambda k_t + z_{\lambda,t}, \quad \tau_t = \theta_\tau k_t + z_{\tau,t}. \quad (14)$$

Constant Balance and Operating Commitment use scalar first-order dynamic laws such as AR(1) or AR(1) in logs. Scale and Balance lets $z_t = (z_{\lambda,t}, z_{\tau,t})'$ follow a VAR(1)

$$z_{t+1} = \mu + A(z_t - \mu) + \varepsilon_{t+1}, \quad \mathbb{E}_t[\varepsilon_{t+1}] = 0. \quad (15)$$

Here $z_{\lambda,t}$ and $z_{\tau,t}$ are the capital-adjusted scale and balance states. This preserves the revenue-cost information while partialing out capital. Their joint law carries both economic coordinates through time. Section 4 estimates that law directly and measures the innovation carried by balance.

2.3 Common dynamic environment

I hold the financial and investment structure common so that the comparison isolates the consequences of the operating technologies. Time is annual, owners are risk neutral, and the firm operates in partial equilibrium. At the start of period t , the firm observes installed

capital K_t and operating state ζ_t . This is beginning capital; empirically, it is capital reported at the preceding fiscal-year end.

A continuing firm earns economic income $\Pi(K_t, \zeta_t)$ and chooses K_{t+1} . Capital obeys

$$K_{t+1} = (1 - \delta)K_t + I_t, \quad \frac{I_t}{K_t} = \frac{K_{t+1}}{K_t} - (1 - \delta), \quad (16)$$

where I_t is gross investment and δ is depreciation. Net capital percent growth is $g_t = K_{t+1}/K_t - 1$. A continuing firm trades capital at unit price and pays the symmetric quadratic adjustment cost

$$\Psi(K_t, K_{t+1}) = \frac{\phi}{2} \left(\frac{K_{t+1}}{K_t} - 1 \right)^2 K_t = \frac{\phi}{2} g_t^2 K_t. \quad (17)$$

The common reversible technology lets firms expand or contract while isolating differences in the operating state. The quadratic cost smoothly moderates expansion and contraction and supplies the conventional local investment response to marginal q ([Abel and Eberly, 1996](#)).

The payout from continuation is

$$D(K_t, K_{t+1}, \zeta_t) = \Pi(K_t, \zeta_t) - [K_{t+1} - (1 - \delta)K_t] - \Psi(K_t, K_{t+1}), \quad (18)$$

and, with $\beta = (1 + r)^{-1}$, continuation value is

$$V^c(K, \zeta) = \max_{K' > 0} \{ D(K, K', \zeta) + \beta \mathbb{E}[V(K', \zeta') \mid \zeta] \}. \quad (19)$$

Losses matter for both continuation policy and survival. Liquidation is chosen before current production and recovers ℓK , with $0 < \ell \leq 1$. A firm with continuation value lower than liquidation value exits. For computation, I replace the maximum of continuation and liquidation values with a regularized comparison

$$V(K, \zeta) = \kappa K \log \left[\exp \left\{ \frac{V^c(K, \zeta)}{\kappa K} \right\} + \exp \left\{ \frac{\ell K}{\kappa K} \right\} \right]. \quad (20)$$

where κ is a small fixed numerical regularization coefficient without economic content. The associated smooth liquidation probability is $[1 + \exp\{(V^c - \ell K)/(\kappa K)\}]^{-1}$. Section 4 describes the numerical implementation and the replacement-entry closure used to construct stationary panels.

For an interior continuing choice, define discounted marginal continuation value as

$$q_t \equiv \beta \mathbb{E}[V_K(K_{t+1}, \zeta_{t+1}) \mid \zeta_t].$$

The first-order condition is

$$q_t = 1 + \phi \left(\frac{K_{t+1}}{K_t} - 1 \right) = 1 + \phi g_t. \quad (21)$$

Thus ϕ governs the local investment response to marginal q . Marginal q and observable average q can differ because the operating technologies, such as Scale and Balance, need not satisfy the degree-one homogeneity conditions in Hayashi (1982).

2.4 Empirical implications

The hierarchy makes three empirical predictions. First, one-factor models restrict balance to a single value or a line/manifold. Second, balance left outside that surface should persist within firms and organize policies, value, and downside outcomes *if it is an economically relevant state*. Third, stochastic balance should change signed-income support, tail behavior, and equilibrium outcomes. Section 3 tests the first two predictions with an outcome-free surface and same-income comparisons. Section 4 estimates the dynamic laws of the models and tests the predictions in the calibrated models.

3 Scale and balance in the data

Proposition 1 makes the conventional one-factor operating architecture observable. When scale intensity orders one smooth firm-specific operating factor, balance must satisfy $\tau_t = h(k_t, x_t)$, with $x_t = \lambda_t - k_t$. I first give that restriction a flexible-form implementation and measure the conditional balance width left around it. I then ask whether this missing margin persists, organizes subsequent corporate outcomes, and remains active after current income and capital are held nearly fixed.

3.1 Data, timing, and operating coordinates

I use 142,480 firm-years for 13,904 nonfinancial public firms from 1971 through 2019. Revenue, cost, and capital are positive where their logarithms enter. Dollar quantities are measured as shares of nominal GDP and expressed in a 2019-sized economy. Appendix Table 8 reports the screens, source variables, accounting identities, and outcome timing.

I use three income definitions. Operating income defines cost as COGS plus SG&A. Gross income uses only COGS as cost. Economic income is defined as net investment retained by the firm plus distributions to debt and equity, using the firm’s sources-and-uses identity. Its corresponding cost is revenue less economic income. For each income definition j , I construct

$$\lambda_t^j = \frac{\log R_t + \log C_t^j}{2}, \quad \tau_t^j = \log R_t - \log C_t^j, \quad x_t^j = \lambda_t^j - k_t.$$

Each definition satisfies $\Pi_t^j = 2e^{\lambda_t^j} \sinh(\tau_t^j/2)$ observation by observation. The surface is re-estimated within each income definition. Adverse CRSP exit is a terminal public-panel row matched to a CRSP delisting code in the 4xx or 5xx class.

3.2 The flexible one-factor surface

I begin with two economic rather than flexible benchmarks. Constant Balance fixes $\tau = -\log \gamma$ at the training-sample median within each cross-fitting fold. Operating Commitment

estimates the cost relation $C = \gamma R + fK$ from observed costs and evaluates the implied analytic manifold in Equation (10). Both retain the assumption of one stochastic operating factor.

I also conduct the central test which gives the ordered one-factor model an arbitrary smooth graph in capital and scale intensity. Its conditional-median specification is¹

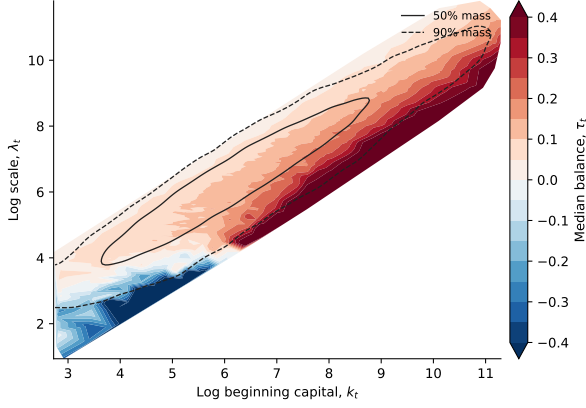
$$\tau_{it} = a_t + h(k_{it}, x_{it}) + u_{it}^{1F}. \quad (22)$$

The function h is a tensor product of flexible cubic B-splines in the observed log coordinates. I select the smallest basis whose mean absolute balance error lies within 0.0005 of the minimum, yielding six spline basis functions per coordinate. Every reported u^{1F} is therefore a measure of balance left around a surface selected solely from current operating coordinates.

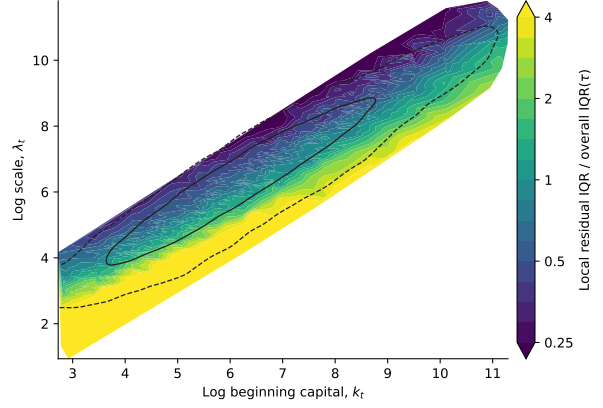
Flexibility improves the conditional center but leaves nearly all of the width. Table 1 reports mean absolute error of 0.161 and residual IQR of 0.143 for Constant Balance. Operating Commitment bends the conditional center but leaves 105.4 percent of unconditional balance IQR, i.e., it worsens the problem. The homogeneous curve $h(x)$ lowers mean absolute error and residual IQR to 0.152 and 0.118. The general $h(k, x)$ lowers mean absolute error further to 0.140, while residual IQR remains 0.118. The best selected one-factor flexible surface therefore leaves 82.3 percent of unconditional balance IQR unexplained. This is the economic content “left on the table” with one operating factor.

Figure 1 shows where that width lives. Capital and scale are tightly related over the populated support. The surface follows conditional median balance, while broad distributions remain around it for both small and large firms. Within annual 10×10 grids in (k, x) , the median local IQR of u^{1F} is 0.104, or 73.1 percent of unconditional balance IQR. Cells containing both profits and losses account for 74.4 percent of covered firm-years. The width is therefore local and densely populated: firms occupy a region around the best one-factor

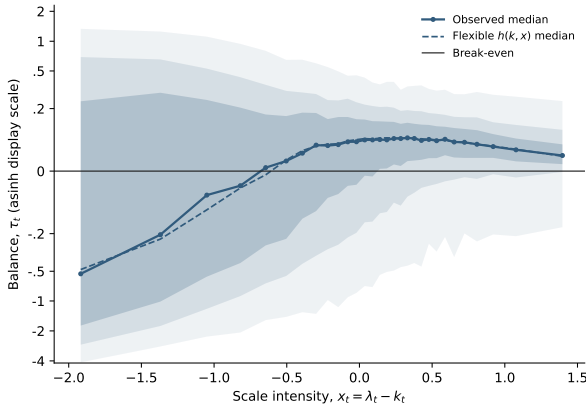
¹The conditional median follows the quantile-regression tradition of [Koenker and Bassett \(1978\)](#); the spline basis belongs to the series-estimation family studied by [Newey \(1997\)](#).



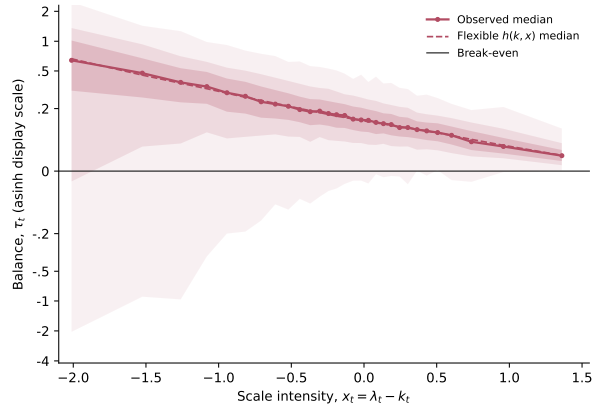
(a) Conditional median balance



(b) Relative residual width



(c) Small firms: lowest capital tercile



(d) Large firms: highest capital tercile

Fig. 1. Conditional center and width in the observable position space. Panels (a) and (b) pool operating-layer firm-years into 30×30 quantile cells formed in beginning capital k and scale intensity $x = \lambda - k$, then display those cells in (k, λ) coordinates. Black contours enclose 50 and 90 percent of firm-years within the plotted support. Panel (a) reports conditional median balance τ . Panel (b) reports the local IQR of the cross-fitted residual $u^{1F} = \tau - h(k, x)$ divided by the unconditional IQR of τ . Its color scale runs from one-quarter to four times the unconditional IQR. Panels (c) and (d) retain the lowest and highest beginning-capital terciles and form 30 equal-population x bins within each. Solid lines and markers report observed median τ ; dashed lines report cross-fitted surface medians. From darkest to lightest, the bands contain the 25th–75th, 10th–90th, and 1st–99th percentiles of τ .

Table 1
Fit, local width, and persistence under one-factor balance restrictions

<i>Panel A. Balance fit</i>			
Restriction	MAE in τ	Residual IQR	IQR remaining (%)
Constant Balance	0.161	0.143	100.0
Operating Commitment	0.165	0.150	105.4
Flexible $h(x)$	0.152	0.118	82.8
Flexible $h(k, x)$	0.140	0.118	82.3

<i>Panel B. Width and persistence after flexible $h(k, x)$</i>		
Statistic	Estimate	NW t -stat.
Median local IQR of u^{1F}	0.104	
Local IQR / IQR(τ) (%)	73.1	
Firm-years in cells with profits and losses (%)	74.4	
Persistence of u^{1F}	0.788	56.12
Persistence after firm centering	0.510	19.79

The table asks whether beginning capital k and one operating factor $x = \lambda - k$ span revenue-cost balance τ . Panel A reports out-of-fold vertical errors. Constant Balance fixes balance at its training-sample median. Operating Commitment fits its parameters to log costs and then evaluates the implied balance curve. The flexible rows estimate outcome-free smooth surfaces; “IQR remaining” is $\text{IQR}(u)/\text{IQR}(\tau)$. Panel B uses $u^{1F} = \tau - h(k, x)$. Local statistics come from annual 10×10 grids in (k, x) , retaining cells with at least 10 firms. Persistence entries are time-series means of annual coefficients from regressions of u_{t+1}^{1F} on u_t^{1F} and (k, x) ; Newey–West(3) t statistics are reported. Firm centering subtracts each firm’s time-series median. The sample contains 142,480 operating-layer firm-years.

graph. This conditional width is the paper’s central stylized fact.

3.3 Persistence and future corporate outcomes

Balance left outside the surface persists within firms. I estimate annual cross-sectional regressions

$$u_{i,t+1}^{1F} = b_t u_{it}^{1F} + m_t(k_{it}, x_{it}) + e_{i,t+1}, \quad (23)$$

where m_t is a flexible functional over the (k, x) basis. The time-series mean of b_t is 0.788, with a Newey–West(3) t statistic of 56.12. Subtracting each firm’s time-series median leaves persistence of 0.510, with a t statistic of 19.79. Balance around the surface thus moves persistently within firms as well as across them.

Table 2
The balance residual and corporate outcomes

<i>Panel A. One-year corporate outcomes</i>				
Outcome	Conditional coefficient	Low	High	High-low
Next-year income / capital	0.362 (14.45)	0.007	0.205	0.198 (21.42)
Next-year investment / capital	0.171 (3.88)	0.021	0.143	0.122 (11.80)
Next-year capital growth	0.178 (4.88)	-0.063	0.060	0.123 (13.04)
Next-year value-to-capital	0.004 (0.02)	1.615	2.029	0.413 (14.63)
Loss next year (%)	-44.7 (20.78)	28.8	3.7	-25.2 (10.84)
Adverse CRSP exit (%)	-4.5 (6.69)	4.3	0.9	-3.5 (6.45)
<i>Panel B. Loss resolution over three years</i>				
Recovery by year 3 (%)	29.1 (11.65)	29.9	59.5	29.6 (14.06)
Persistent loss by year 3 (%)	-14.9 (3.05)	33.8	14.0	-19.8 (8.47)
Persistent loss or adverse CRSP exit (%)	-31.9 (12.27)	65.7	34.8	-30.9 (16.14)

The balance residual is $u^{1F} = \tau - h(k, x)$, where $x = \lambda - k$. Conditional coefficients are annual coefficients on u^{1F} with flexible controls for (k, x) ; Low and High are residual terciles within annual (k, x) cells. Panel A uses 10×10 cells in the full sample. Panel B uses 5×5 cells among current-loss firms. Next income and investment are divided by K_t , next capital growth is $\log K_{t+1} - \log K_t$, and next value-to-capital is V_{t+1}/K_{t+1} . Binary entries are percentages or percentage points. Absolute Newey–West(3) t statistics appear in parentheses.

I next ask whether the balance residual organizes corporate outcomes. For outcomes with an exact consecutive-year lead, I estimate

$$Y_{i,t+1} = b_t^Y u_{it}^{1F} + m_t^Y(k_{it}, x_{it}) + e_{it}^Y. \quad (24)$$

with flexible conditioning on (k, x) ; I fix the residuals u_{it}^{1F} before examining these forward associations.

Table 2 shows a sharp corporate-outcome gradient. Moving from the low to the high balance-residual tercile raises next-year income, investment, capital growth, and closing value-to-capital by 0.198, 0.122, 0.123, and 0.413. Next-year loss falls by 25.2 percentage points and adverse CRSP exit by 3.5 points. The investment, capital-growth, value, and event results show that the missing margin is economically active beyond persistence in the accounting outcome. For value, the conditional linear coefficient is near zero, while the within-cell high–low spread is 0.413; the endpoint comparison carries the value result.

Among current-loss firms, the same ordering governs three-year resolution. Recovery is

29.6 percentage points more common in the high balance-residual tercile, persistent loss is 19.8 points less common, and persistent loss or an adverse CRSP exit is 30.9 points less common. Balance therefore organizes both near-term policy and the subsequent path out of loss. This ordering connects to evidence separating transitory from persistent losses and their value and abandonment implications (Hayn, 1995; Joos and Plesko, 2005), persistent negative cash flow and corporate policy (Denis and McKeon, 2021), and the changing demand-building investment and cost composition of loss firms (Gosselin and Zhang, 2026).

3.4 Same income, different revenue-cost composition

Current income is the most direct alternative scalar summary. I compare firms near the 25th, 50th, and 75th percentiles of absolute income separately among losses and profits. Within each fiscal year and beginning-capital decile, firms near a given income contour are split into low and high $x = \lambda - k$ groups. The endpoints have nearly identical current income and capital but substantially different scale intensity x .

Table 3

Revenue-cost composition and corporate outcomes at the same current income and capital

<i>Panel A. Current outcomes</i>													
Income group	Investment / capital			Capital growth			Value-to-capital			Current balance τ_t			$\Delta (t)$
	Low	High	$\Delta (t)$	Low	High	$\Delta (t)$	Low	High	$\Delta (t)$	Low	High	$\Delta (t)$	
Large loss	-0.045	0.085	0.131 (3.37)	-0.174	-0.088	0.086 (2.51)	2.703	2.755	0.053 (0.24)	-0.965	-0.383	0.582 (24.96)	
Median loss	-0.064	0.024	0.088 (2.58)	-0.197	-0.106	0.091 (3.82)	2.296	2.412	0.116 (0.75)	-0.439	-0.166	0.272 (12.82)	
Small loss	-0.083	0.054	0.136 (4.80)	-0.209	-0.080	0.129 (6.79)	2.274	2.309	0.035 (0.19)	-0.245	-0.090	0.155 (7.21)	
Small profit	0.081	0.116	0.035 (3.36)	0.006	0.043	0.036 (4.12)	1.575	1.429	-0.147 (3.23)	0.205	0.082	-0.123 (19.67)	
Median profit	0.107	0.133	0.026 (2.89)	0.034	0.059	0.025 (3.94)	1.732	1.571	-0.161 (5.87)	0.261	0.097	-0.164 (14.89)	
Large profit	0.096	0.112	0.015 (1.86)	0.029	0.046	0.017 (2.78)	1.842	1.708	-0.134 (3.02)	0.329	0.118	-0.211 (17.61)	

<i>Panel B. Next-year outcomes</i>													
Income group	Investment / capital			Capital growth			Value-to-capital			Loss (%)			$\Delta (t)$
	Low	High	$\Delta (t)$	Low	High	$\Delta (t)$	Low	High	$\Delta (t)$	Low	High	$\Delta (t)$	
Large loss	0.032	0.094	0.062 (1.33)	-0.111	-0.072	0.039 (1.06)	2.990	2.945	-0.044 (0.16)	87.0	78.7	-8.3 (3.12)	
Median loss	0.034	0.031	-0.003 (0.09)	-0.112	-0.096	0.017 (0.47)	2.418	2.344	-0.073 (0.46)	73.2	66.5	-6.7 (1.49)	
Small loss	0.070	0.093	0.024 (0.35)	-0.079	-0.048	0.031 (0.73)	2.553	2.329	-0.224 (1.84)	66.9	56.3	-10.7 (3.48)	
Small profit	0.089	0.093	0.004 (0.70)	0.012	0.026	0.014 (2.33)	1.571	1.412	-0.159 (3.06)	6.2	6.3	0.1 (0.17)	
Median profit	0.097	0.100	0.002 (0.38)	0.025	0.036	0.011 (2.29)	1.703	1.544	-0.159 (4.61)	3.0	2.2	-0.8 (1.81)	
Large profit	0.090	0.086	-0.004 (0.79)	0.022	0.024	0.002 (0.36)	1.798	1.667	-0.132 (3.90)	1.2	1.0	-0.2 (0.72)	

The six income groups are centered at approximately $-\$34m$, $-\$12m$, $-\$4m$, $\$25m$, $\$103m$, and $\$448m$, respectively, in millions of dollars on the paper's nominal-GDP-to-2019 scale. Each group contains firms within 0.20 log points of its target absolute income. Within fiscal year and beginning-capital decile, firms are split at median scale intensity, $x_t = \lambda_t - \log K_{t-1}$. Panel A reports I_t/K_{t-1} , $\log K_t - \log K_{t-1}$, V_t/K_t , and τ_t . Panel B reports the exact next-year counterparts, with loss replacing balance: I_{t+1}/K_t , $\log K_{t+1} - \log K_t$, V_{t+1}/K_{t+1} , and an indicator for $\Pi_{t+1} < 0$. The maximum absolute high-minus-low gap is 0.010 log points for income and 0.030 log points for capital; the corresponding x_t gap ranges from 0.886 to 1.083. Entries are annual equal-weighted averages. Loss is in percent and its Δ is in percentage points; all other entries are in natural units. Each Δ is High minus Low. The last column in each block reports the absolute Newey–West t -statistic in parentheses.

Table 3 shows the current and next-year differences. Panel A first quantifies the observable positions being compared. At fixed income, the signed-income identity $\Pi = 2e^\lambda \sinh(\tau/2)$ requires greater scale to move balance toward zero. Among loss firms, the high- x endpoint has a balance that is 0.155 to 0.582 higher and therefore less negative. Among profitable firms, its balance is 0.123 to 0.211 lower and therefore thinner. The balance block describes the geometry of the matched comparison: it is the operating composition hidden by current income.

That composition is economically active. At all three loss contours, low- x firms disinvest while high- x firms invest. High-minus-low investment differences range from 0.088 to 0.136 of beginning capital, and capital growth is 0.086 to 0.129 log points higher. Among profitable firms, both differences remain positive but smaller. Current value-to-capital is similar across the loss endpoints, whereas the high- x , low- τ profitable firms have value-to-capital that is 0.134 to 0.161 lower. The same current income and capital therefore support different policies and valuations when their revenue-cost compositions differ.

Panel B shows what persists. The large contemporaneous investment differences among loss firms largely disappear from next-year investment, consistent with a policy response to the current operating position rather than a permanent investment type. Among profitable firms, the high- x endpoint continues to grow capital somewhat faster at the small and median profit contours, while its lower value-to-capital persists at all three contours. The clearest loss result is resolution: among firms currently reporting losses, the high- x endpoint is 6.7 to 10.7 percentage points less likely to report another loss next year. The difference is strongest at the small- and large-loss contours. Once firms are profitable, next-year loss is uncommon at both endpoints.

Investment depends on the return to the next unit of capital and on the distribution of continuation payoffs, rather than on current income alone. Equation (13) displays the first channel. At fixed Π/K , the marginal operating return can still vary with the operating base $(R + C)/K$ when revenue and cost have different capital loadings. Persistence in scale

and balance supplies the broader continuation channel: current composition marks different positions in the state system and hence different future revenue-cost distributions. For a firm currently making a loss, the high- x position is a large revenue-and-cost operation closer to break-even and more likely to recover. For a profitable firm, it is a high-volume, thin-margin operation with lower current and next-year value relative to capital. Scale is not uniformly favorable or unfavorable; its implications depend on the revenue-cost wedge that supports it.

Income and capital therefore leave economically active revenue-cost composition unspanned. Firms that a scalar income-capital description treats as alike make different capital choices, receive different valuations, and follow different paths out of loss. Together with the persistence and subsequent-outcome evidence, the same-income comparison earns the interpretation of balance as a second operating state rather than a relabeling of current income.

3.5 Stability and empirical conclusion

The result travels across accounting boundaries and temporal aggregation. I fit the surface separately within each income boundary and rebuild it over exact consecutive three-year windows. Appendix Table 9 reports rank correlation 0.774 between operating- and economic-income residuals and sign agreement in 82.4 percent of common firm-years. Under exact three-year means, residual IQR remains 0.781 and 0.803 of balance IQR at the operating- and economic-income boundaries.

Balance can reflect moving differences in technology, organization, markups, product appeal, or intangible intensity. Whatever its economic source, the observable coordinate changes within firms, persists after firm centering, and organizes policy and payoffs conditional on capital and scale intensity. Balance gaps are clearly *caused* by other, deeper factors, but balance is an observable economically-active state of the firm that is currently being ignored.

The empirical conclusion is direct. The best flexible surface afforded to a firm model with capital and one regular operating factor leaves a locally wide and persistent balance coordinate. That coordinate predicts future corporate outcomes conditional on capital and

scale intensity and remains distinct from current income. Section 4 asks what changes when q-theory carries that missing margin.

4 Scale and balance in q-theory

Section 3 shows that firms retain a persistent and payoff-relevant balance margin outside the best ordered-factor surface. I now ask what changes when q-theory carries it. I hold the financial environment fixed and compare three operating technologies: one fixes balance, one traces a deterministic revenue-cost manifold using “fixed” costs proportional to capital, and one lets scale and balance move separately.

Four results emerge. The solved economies display the fixed balance, manifold, and region implied by their technologies. Stochastic balance generates much of the observed income-to-capital dispersion, loss rates, and transformed-income-change tails under Gaussian innovations. It also reproduces the empirical directions of investment, value, loss, and liquidation across the balance margin. Freezing balance within the same solved economy removes those outcomes. Direct estimates anchor the scale–balance law and reveal two distinct reduced-form innovations, further supporting the “two states” hypothesis.

4.1 Common environment, parameters, and calibration

I hold the financial environment common across models. Every economy uses the capital law in Equation (16), $r = 0.04$, $\delta = 0.042$, proportional liquidation recovery $\ell = 0.80$, and the symmetric adjustment cost functional form in Equation (17). Sample discount-rate evidence motivates r (e.g. median interest/debt), while median annual investment relative to capital anchors depreciation. Recovery fixes the terminal liquidation technology across models at an empirically plausible rate of 80%.

The operating parameters determine what each technology can represent. Constant Balance fixes balance at its empirical median, $\bar{\tau} = 0.0641$, and uses $\theta_\lambda = 0.80$, a loading in the

tradition of [Hennessy and Whited \(2007\)](#); the fixed balance implies $\gamma = \exp(-\bar{\tau})$. Operating Commitment is evaluated at $\theta_\lambda = 0.80$ and 0.94 . Both versions use the directly measured expense-to-capital commitment $f = 0.0206$, which implies variable-cost shares $\gamma = 0.9100$ and 0.9162 . Scale and Balance uses $(\theta_\lambda, \theta_\tau) = (0.94, -0.002)$. The empirical scale-capital relation supplies the starting value for θ_λ ; I evaluate nearby loading values, including the small balance loading, against central state and finance outcomes. The columns therefore separate the conventional one-factor loading, an empirically calibrated one-factor manifold, and a stochastic scale-balance region.

I estimate the operating dynamics from 124,842 consecutive beginning-capital firm-year pairs. The one-factor economies retain scalar AR(1) persistence and innovation scales from their corresponding benchmark estimates. The estimated AR(1) locations perform poorly in the one-factor economies, so I allow them the extra flexibility of estimating location corrections. Their location corrections are 0.2125 for Constant Balance, 0.4500 for Operating Commitment at the 0.80 loading, and 0.2500 for Operating Commitment at the 0.94 loading. These corrections recenter equilibrium scale and capital without changing the persistence or innovation scale of the inherited scalar law. [Table 4](#) separately reports the directly estimated contribution-margin counterpart for Operating Commitment at the 0.94 loading, so that its empirical counterpart and the scalar model input remain visible.

Scale and Balance instead uses the joint VAR in Equation [\(15\)](#). I estimate its transition coefficients on median-centered states. Its directly estimated locations, transitions, innovation scales, and correlation enter the structural model without correction. The direct joint law remains compatible with optimal investment, liquidation, replacement entry, and the resulting stationary economy without any recentering.

With the technologies and state laws fixed, I choose symmetric adjustment curvatures of $\phi = 9.25, 11.75, 4.75$, and 5.50 for Constant Balance, Operating Commitment at the 0.80 and 0.94 loadings, and Scale and Balance. I use investment– $\log(V/K)$ slope, together with central capital, investment, and value levels, to choose these curvatures. Balance width, loss

rates, transformed-income-change tails, severe losses, liquidation, and the Section 3 outcome gradients don't enter or impact those choices.

All models use a fixed logistic scale, $\kappa = 0.01$, which smooths the Bellman choice without adding an economic margin. Replacement entry draws one firm from the next-period survivor distribution for every liquidation, preserving firm mass. Every problem uses 1,001 log-capital nodes over $[2, 12]$. The one-factor economies use 15 Gaussian state nodes; Scale and Balance uses 15 scale nodes and nine balance nodes.² I validate stationary cross-sections from the invariant distributions and use six independent continuous panels, with interpolated values, policies, and Bellman gaps, for simulated outcomes.

4.2 Scale and balance move separately

The data directly determine the dynamics of scale and balance, conditional on $(\theta_\lambda, \theta_\tau)$. At the Scale and Balance loading, the two capital-adjusted states are

$$z_{\lambda,t} = \lambda_t - 0.94k_t, \quad z_{\tau,t} = \tau_t + 0.002k_t.$$

Here z_λ is the capital-adjusted scale state and z_τ is the capital-adjusted balance state. Table 4 reports direct data estimates, model inputs, and dynamic laws estimated in continuous simulations. Scale is more persistent than balance: the own-state coefficients are 0.9266 and 0.6044. The cross-lag coefficients are economically small but distinguishable from zero, -0.0229 (0.0054) from balance to scale and 0.0086 (0.0027) from scale to balance. Re-estimation in the simulated panels closely reproduces the transition coefficients and innovation scales entered into the model, indicating that the model directly follows the dynamic law it observes.

At the reduced-form level, scale and balance also receive distinct contemporaneous innovations. Their correlation is 0.014. A single additive contemporaneous innovation instead

²The Gaussian transition cells follow the finite-state Markov approximation family of Tauchen (1986).

implies absolute correlation one ([Funovits and Braumann, 2021](#)). The fitted state law therefore contains a nondegenerate balance innovation largely orthogonal to the scale innovation.

A structural decomposition into demand, technology, markup, organization, or cost shocks would require additional restrictions ([Kilian and Lütkepohl, 2017](#)). The separate reduced-form movement is economically relevant: corporate-finance models with multiple cash-flow components likewise show that shock composition changes policy and value ([Gorbenko and Strebulaev, 2010](#)). I use Gaussian innovations in every calibrated economy to isolate the consequence of carrying the balance margin from the consequence of heavy-tailed state transitions. Empirical scale–balance innovations are heavier tailed, and non-Gaussian firm transitions can add materially to value and selection ([Berlingieri, Jaimovich, Terry, and Vincent, 2025](#)).

Table 4
Model-native state laws and the Scale and Balance innovation dimension.

State-law object	Direct data estimate	Model input	Simulation	Diff.
<i>Panel A: Direct counterparts, model inputs, and simulated laws</i>				
<i>OC (.94): Contribution-margin state</i>				
Location, μ	-1.9517 (.0065)	-2.0079	-1.9275	.0242
Own-state transition	.9120 (.0022)	.9265	.9266	.0145
Innovation scale	.1599 (.0013)	.1498	.1466	-.0132
<i>SB (.94): Capital-adjusted scale state</i>				
Location, μ_λ	.4885 (.0071)	.4885	.5653	.0768
Own-state transition, $A_{\lambda\lambda}$	.9266 (.0017)	.9266	.9267	.0001
Innovation scale, $\sigma_{\varepsilon_\lambda}$	.1469 (.0011)	.1469	.1437	-.0032
<i>SB (.94): Capital-adjusted balance state</i>				
Location, μ_τ	.0774 (.0008)	.0774	.0819	.0045
Own-state transition, $A_{\tau\tau}$	.6044 (.0098)	.6044	.6031	-.0012
Innovation scale, $\sigma_{\varepsilon_\tau}$	.0553 (.0013)	.0553	.0540	-.0013
<i>Panel B: Scale–balance cross-dynamics and innovation dimension</i>				
SB joint-law object	Direct data estimate	SB simulation	Diff.	
$A_{\lambda\tau}$	-.0229 (.0054)	-.0283	-.0055	
$A_{\tau\lambda}$	.0086 (.0027)	.0085	-.0001	
Innovation correlation, $\rho_{\lambda,\tau}$	.0140 (.0064)	.0122	-.0018	
Innovation eigenvalue ratio, $\lambda_{\min}/\lambda_{\max}$	.1417 (.0062)	.1411	-.0006	
Balance variance unspanned, $1 - \rho_{\lambda,\tau}^2$	.9998 (.0002)	.9999	.0000	

Notes: OC is Operating Commitment and SB is Scale and Balance. OC’s contribution-margin state is $z_t^C = r_t - .94 \log K_{t-1} + \log(1 - \gamma)$, with $\gamma = .9162$; it is the log revenue contribution available before the fixed operating commitment. The direct OC estimates apply this definition to observed revenue and beginning capital. The promoted OC model instead inherits the theta-.94 scalar parent law: its parent location is -2.2579, its location correction is .2500, and its resulting input location is -2.0079; its transition and innovation scale are inherited without correction. SB uses $z_{\lambda,t} = \lambda_t - .94 \log K_{t-1}$ and $z_{\tau,t} = \tau_t + .002 \log K_{t-1}$. Direct estimates use 124,842 consecutive beginning-capital firm-year pairs. Firm-history bootstrap standard errors from 999 draws appear in parentheses beside direct estimates. Location corrections apply only to state locations. SB uses its direct estimates as model inputs without correction. Innovation scales equal the coordinatewise 1–99-percent-trimmed residual IQR divided by the Normal IQR. In A_{ij} , current state j predicts next-period state i . Simulation entries re-estimate the same model-native law on six continuous simulated panels and report their mean. Diff. is simulation minus the direct data counterpart.

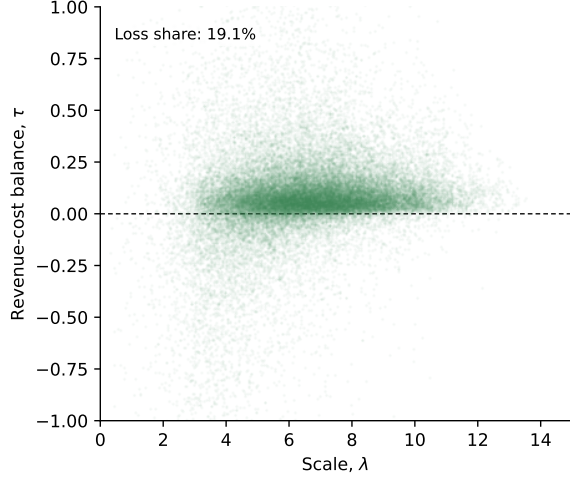
4.3 From fixed balance to a manifold to a region

Figure 2 places the theoretical restrictions next to the data. Panels A, B, and D use (λ, τ) coordinates. Panel C uses $x = \lambda - \log K$ because Operating Commitment occupies the analytic manifold $\tau = h_f(x)$.

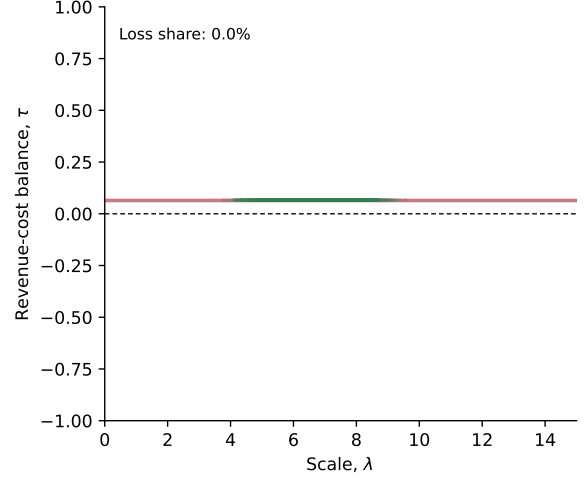
The equilibrium distributions display the theory directly. Constant Balance concentrates at its calibrated balance value. Operating Commitment follows its deterministic manifold, bending toward break-even as the capital commitment grows relative to the scale of revenue and cost. Scale and Balance spans a two-dimensional region containing both profits and losses. One stochastic operating factor therefore fixes balance or traces a curve; stochastic balance fills the region between revenue and cost.

Table 5 applies the same estimators to the data and six continuous panels from each model. The common environment keeps the economies in a recognizable corporate-finance range. Median log capital lies within 0.18 of the data in every column. Median income-to-capital ranges from 0.0775 to 0.0955, versus 0.0816 in the data, and median scale is close in all four models. Median investment is close to the 0.0438 data value as well.

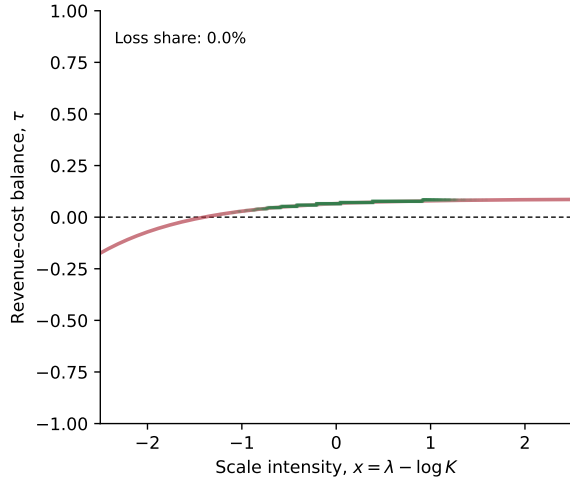
Signed outcomes separate the technologies. Constant Balance has no balance width. The Operating Commitment economies have balance IQR below 0.009 and place essentially no equilibrium mass below break-even. Scale and Balance produces balance IQR 0.0927 versus 0.1151 in the data, income-to-capital IQR 0.1160 versus 0.1089, loss rate 15.74 versus 19.09 percent, and income-change tail rate 21.23 versus 20.12 percent. Gaussian balance dynamics therefore supply most of the observed balance width, income-to-capital dispersion, loss rate, and income-change tail rate. The remaining left-tail gap is severity: the model's severe-loss rate is 1.51 percent versus 9.14 percent in the data.



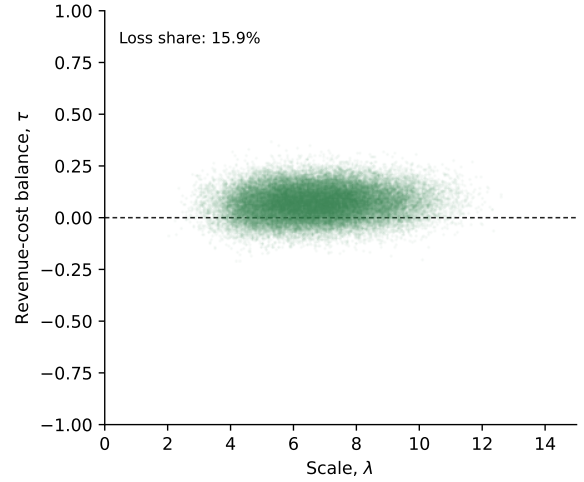
(a) Data



(b) Constant Balance (.80)



(c) Operating Commitment (.94)



(d) Scale and Balance (.94)

Fig. 2. Operating states in the data and calibrated models. Panels A, B, and D plot balance against scale, λ . Panel C instead uses scale intensity, $x = \lambda - \log K$, because the Operating Commitment model's one-factor restriction is the analytic manifold $\tau = h(x)$. The dashed line marks break-even, $\tau = 0$. The red lines mark Constant Balance and the Operating Commitment model's analytic manifold; simulated and data observations are drawn as green scatters.

Table 5
Calibrated outcomes across the operating-state model progression.

Outcome	Data	CB (.80)	OC (.80)	OC (.94)	SB (.94)
<i>Panel A: Cross-sectional states and economic outcomes</i>					
Scale, median λ	6.436	6.533	6.469	6.414	6.651
Scale, IQR λ	2.793	1.049	1.048	2.084	2.158
Balance, median τ	0.0641	0.0641	0.0769	0.0687	0.0688
Balance, IQR τ	0.1151	0.0000	0.0086	0.0087	0.0927
Capital, median $\log K_{t-1}$	6.344	6.164	6.248	6.253	6.452
Value-to-capital, median V_t/K_t	1.285	1.598	1.733	1.273	1.395
Value-to-capital, IQR V_t/K_t	0.923	0.530	0.663	0.505	0.642
Income-to-capital, median Π_t/K_{t-1}	0.0816	0.0926	0.0955	0.0789	0.0775
Income-to-capital, IQR Π_t/K_{t-1}	0.1089	0.0465	0.0580	0.0467	0.1160
Investment, median I_t/K_{t-1}	0.0438	0.0399	0.0400	0.0420	0.0514
Investment- $\log(V/K)$ slope	0.1720	0.0921	0.0775	0.1628	0.1432
Investment dispersion, IQR I_t/K_{t-1}	0.1934	0.0310	0.0306	0.0624	0.0620
<i>Panel B: Operating-state dynamics and history-dependent outcomes</i>					
Scale change, median $\Delta\lambda$	0.0119	-0.0001	-0.0004	0.0014	0.0113
Scale change, IQR $\Delta\lambda$	0.1919	0.1897	0.1879	0.2035	0.2019
Balance change, median $\Delta\tau$	-0.0002	0.0000	0.0000	-0.0001	-0.0018
Balance change, IQR $\Delta\tau$	0.0719	0.0000	0.0032	0.0037	0.0830
Income-to-capital rank persistence	0.5557	0.9180	0.9171	0.8905	0.5986
Investment persistence	0.1446	0.9056	0.9044	0.8784	0.8332
Loss rate (percent)	19.09	0.00	0.00	0.00	15.74
Loss persistence (percent)	53.23	N/A	N/A	N/A	45.46
Severe-loss rate (percent)	9.14	0.00	0.00	0.00	1.51
Transformed-income-change tail rate (percent)	20.12	0.00	0.00	0.00	21.23
Adverse CRSP exit / liquidation rate (percent)	2.34	0.05	0.10	1.09	1.76

Notes: Model entries are means across six independently simulated continuous panels; every entry applies the same estimator to the observed and simulated firm-year panels. Parenthetical labels report θ_λ . CB is Constant Balance, the conventional fixed-balance one-factor model. OC is Operating Commitment, which adds a capital-proportional operating commitment, first at the conventional loading and then at the empirically calibrated loading. SB is Scale and Balance, which replaces the one-factor balance restriction with a stochastic balance state, with $\theta_\lambda = .94$ and $\theta_\tau = -.002$. Other model-specific parameters are calibrated within each environment. K_{t-1} is beginning capital; V_t/K_t is closing average q. State changes, rank persistence, and loss persistence use exact consecutive incumbent observations. Income-to-capital rank persistence is the Spearman correlation between current and next-period income-to-capital. Loss persistence is $\Pr(\Pi_{t+1} < 0 \mid \Pi_t < 0)$; N/A denotes fewer than 100 current-loss origins in every simulation panel. A severe loss is $\Pi_t/K_{t-1} < -0.10$; the transformed-income-change tail is $\Pr\{|\Delta \operatorname{asinh}(\Pi_t)| > 2\}$ among consecutive incumbent observations. Rate rows are percentages. The data rate is adverse CRSP exit; model rates are Bellman liquidation choices.

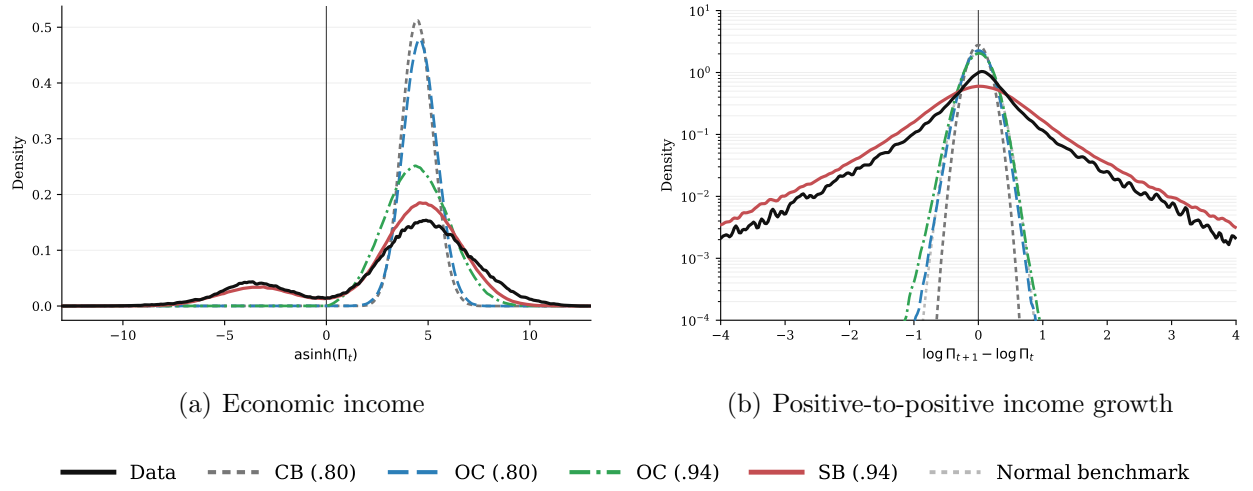


Fig. 3. Economic income and income growth across calibrated models. Panel A plots the density of $\text{asinh}(\Pi_t)$, where Π_t is economic income; the vertical line marks break-even. Panel B plots $\log \Pi_{t+1} - \log \Pi_t$ for exact consecutive incumbent observations with $\Pi_t > 0$ and $\Pi_{t+1} > 0$; negative values are income declines, the vertical line marks no change, and the density axis is logarithmic. CB, OC, and SB denote Constant Balance, Operating Commitment, and Scale and Balance; parenthetical labels report θ_λ .

Figure 3 shows the corresponding full distributions. Constant Balance and Operating Commitment concentrate income above break-even into a log-Normal, and produce narrow, roughly Normal, positive-income growth. Scale and Balance produces signed income and broad growth tails resembling those in the data, with a more dispersed positive-to-positive income growth distribution.

The principal quantitative shortfall lies in the common investment technology. The investment- $\log(V/K)$ slope is 0.172 in the data and equals 0.092, 0.078, 0.163, and 0.143 across the four model columns. Investment IQR is 0.1934 in the data and only 0.0310–0.0624 in the models, while investment persistence is 0.1446 in the data and 0.8332–0.9056 in the models. The smooth reversible adjustment cost produces investment that is too concentrated and persistent; nonconvex and irreversible adjustment margins provide established mechanisms for lumpy investment, inaction, and contractions (Abel and Eberly, 1996; Cooper and Haltiwanger, 2006).

Liquidation remains an untargeted outcome. The data have a 2.34-percent adverse CRSP exit rate, while Bellman liquidation rates are 0.05, 0.10, 1.09, and 1.76 percent across the four model columns. The active liquidation margins in Operating Commitment at the 0.94 loading and Scale and Balance models emerge from solved continuation values rather than calibration to specific termination rates.

4.4 What stochastic balance contributes

The cleanest comparison changes only stochastic balance. Table 6 fixes balance at its empirical median within the Scale and Balance environment and re-solves the economy. Unlike the separate Constant Balance benchmark, this matched counterfactual retains the 0.94 scale loading, the marginal scale law, and every other parameter and numerical choice from Scale and Balance.

Freezing balance eliminates losses, severe losses, and transformed-income-change tails. Restoring the stochastic state raises income-to-capital IQR from 0.0379 to 0.1160, the loss

Table 6
What stochastic balance contributes.

Outcome	Data	Matched CB (.94)	SB (.94)	SB – CB
Scale, median λ	6.436	5.777	6.651	0.873
Scale, IQR λ	2.793	1.701	2.158	0.456
Balance, median τ	0.0641	0.0641	0.0688	0.0047
Balance, IQR τ	0.1151	0.0000	0.0927	0.0927
Capital, median $\log K_t$	6.344	5.542	6.452	0.910
Value-to-capital, median V_t/K_t	1.285	1.177	1.395	0.219
Value-to-capital, IQR V_t/K_t	0.923	0.366	0.642	0.277
Income-to-capital, median Π_t/K_{t-1}	0.0816	0.0802	0.0775	-0.0027
Income-to-capital, IQR Π_t/K_{t-1}	0.1089	0.0379	0.1160	0.0781
Investment, median I_t/K_{t-1}	0.0438	0.0400	0.0514	0.0114
Investment dispersion, IQR I_t/K_{t-1}	0.1934	0.0451	0.0620	0.0169
Investment– $\log(V/K)$ slope	0.1720	0.1553	0.1432	-0.0121
Loss rate (percent)	19.09	0.00	15.74	15.74
Severe-loss rate (percent)	9.14	0.00	1.51	1.51
Transformed-income-change tail rate (percent)	20.12	0.00	21.23	21.23
Adverse CRSP exit / liquidation rate (percent)	2.34	0.86	1.76	0.90

Notes: Matched CB (.94) fixes balance at its empirical median, $\tau = 0.0641$, and sets $\theta_\tau = 0$ in the promoted SB (.94) economy. It retains $\theta_\lambda = .94$, the marginal λ law, all other parameters, and the numerical and simulation design; nothing is retargeted. SB (.94) restores stochastic balance and its joint state law. Model entries average the same six continuous simulations, and SB – CB reports their difference. V_t/K_t is closing average q. Rate rows are percentages; the data event is adverse CRSP exit and model liquidation is the Bellman choice.

rate from zero to 15.74 percent, the transformed-income-change tail rate from zero to 21.23 percent, and liquidation from 0.86 to 1.76 percent. Median income-to-capital changes by only -0.0027 . Scale, capital, and value also adjust through optimal policy and selection. Balance therefore changes downside support and the dispersion of corporate outcomes rather than simply shifting the center of the economy.

4.5 Policy, value, and downside in the model

I next ask whether the balance residual organizes firms inside the model as it does in Section 3. I apply the same fitted surface to a long continuous Scale and Balance panel, form balance-residual portfolios within the same annual (k, x) cells, and leave all model inputs unchanged.

Table 7
Corporate outcomes across the balance residual: data and model

	Data			Model		
<i>Panel A: Outcomes one year later</i>						
Next-year outcome	Low	High	Difference	Low	High	Difference
Income / capital	0.007	0.205	0.198 (21.42)	0.033	0.142	0.109
Investment / capital	0.021	0.143	0.122 (11.80)	0.053	0.073	0.021
Capital growth	-0.063	0.060	0.123 (13.04)	0.009	0.029	0.021
Value-to-capital	1.615	2.029	0.413 (14.63)	1.396	1.635	0.238
Loss (%)	28.8	3.7	-25.2 (10.84)	32.9	2.8	-30.0
Adverse CRSP exit / liquidation (%)	4.3	0.9	-3.5 (6.45)	7.3	0.0	-7.3
<i>Panel B: Loss resolution within three years</i>						
Three-year loss outcome	Low	High	Difference	Low	High	Difference
Recovery	29.9	59.5	29.6 (14.06)	61.5	82.6	21.1
Persistent loss	33.8	14.0	-19.8 (8.47)	10.4	5.7	-4.8
Persistent loss or adverse CRSP exit / liquidation	65.7	34.8	-30.9 (16.14)	38.5	17.4	-21.1

In Panel A, low and high are bottom and top terciles of the balance residual within annual 10×10 cells in beginning capital and scale intensity $x = \lambda - \log K$. Panel B first conditions on firms currently making losses and then forms the same residual terciles within annual 5×5 state cells. The data coordinate is $u^{1F} = \tau - h(k, x)$; the model coordinate is constructed by applying the same cross-fitted surface estimator to a continuous simulation from the Scale and Balance model. All outcomes except the terminal event are exact one-year leads. Next value-to-capital is closing V_{t+1}/K_{t+1} . The data terminal outcome is adverse CRSP exit attached to the current firm-year; its model counterpart is Bellman liquidation. Binary outcomes and their differences are percentage points. Panel B follows the incumbent until recovery, liquidation, or the end of the three-year window. Absolute Newey–West t -statistics appear beside data differences.

Table 7 reproduces all six corporate-outcome directions. Model high-minus-low spreads for investment, capital growth, and value-to-capital are 0.021, 0.021, and 0.238, compared with 0.122, 0.123, and 0.413 in the data. Its loss spread is -30.0 percentage points and its Bellman-liquidation spread is -7.3 points, compared with -25.2 and -3.5 points for next-year loss and adverse CRSP exit. The model therefore reproduces the empirical ordering of policy, value, and downside, while compressing the policy and value magnitudes.

The same ordering governs paths out of loss. Recovery rises by 21.1 percentage points between low and high model balance-residual terciles, versus 29.6 points in the data. The combined probability of persistent loss or liquidation falls by 21.1 points, versus 30.9 points. Persistent loss itself falls by 4.8 points, compared with 19.8 points in the data. Surviving model losses therefore resolve faster than their empirical counterparts, but balance orders recovery and continued distress in the same direction.

The economic mechanism is simple. At a given capital stock, scale determines the size of revenue and cost, while balance determines the wedge between those positive flows. Their joint movement changes current marginal returns through Equation (13), shifts the distribution of future signed payoffs, and moves firms toward or away from the liquidation boundary. Constant Balance fixes that wedge, Operating Commitment traces it along a deterministic manifold, and Scale and Balance lets it move independently. Within an otherwise common q-theory environment, carrying the missing balance margin supplies signed support, downside variation, and the empirical ordering of investment, value, losses, and liquidation.

5 Conclusion

Income is revenue minus cost, so a model of the firm must describe both flows. When capital and one ordered operating factor generate them, revenue-cost balance lies on the observable surface $\tau = h(k, \lambda - k)$. Firms instead occupy a persistently thick region around the best flexible surface: the residual retains 82.3 percent of the unconditional interquartile range of balance. The ordered one-factor operating architecture therefore suppresses an important margin of variation in where firms lie between revenue and cost.

Balance is economically active within that region. It persists within firms and organizes investment, capital growth, value, losses, and adverse CRSP exit conditional on capital and scale intensity. It also distinguishes firms with nearly identical capital and current income. Across the matched loss contours, current investment differs by as much as 0.136 of beginning capital, and firms with larger revenue and cost are 6.7 to 10.7 percentage points less likely to report another loss next year. Current income alone therefore combines operating positions with different policies, valuations, and paths through break-even.

The q-theory comparison supplies the model repair. Constant Balance fixes balance at one value, Operating Commitment traces a deterministic manifold, and Scale and Balance spans a region. The matched counterfactual makes the contribution transparent. Freezing balance

within the solved Scale and Balance economy eliminates losses, severe losses, and transformed-income-change tails and reduces income-to-capital IQR from 0.1160 to 0.0379, while median income-to-capital changes little. Restoring stochastic balance generates downside support and dispersion, raises liquidation, and reproduces the empirical direction of policy, value, and loss resolution. The fitted scale-balance innovation covariance is full rank, with correlation 0.014, so the added coordinate is disciplined by distinct reduced-form movement.

The modeling implication is simple. Revenue-cost composition remains consequential after capital and current income are held fixed: firms at the same capital and income can differ in scale, future signed payoffs, investment policy, and downside exposure. Quantitative corporate-finance models concerned with these outcomes must carry scale and balance, or economic equivalents, as distinct operating states.

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A Proof, construction, and measurement

Proof of the one-factor surface proposition

Let $F(k, z) = \Lambda(k, z) - k$. For each supported k , $F(k, \cdot)$ is one-to-one on the connected interval $\mathcal{Z}(k)$ and has a nonzero derivative. The implicit-function theorem therefore supplies a continuously differentiable local inverse g around every supported point such that

$$x = F(k, z) \iff z = g(k, x).$$

Conditional injectivity makes any two local inverses agree on their overlap, so the local functions paste into one inverse over the supported image. Substituting into the smooth balance function gives $h(k, x) = T(k, g(k, x))$ and hence $\tau = h(k, x)$. The assumption is support-wide, not merely pointwise: it excludes folds and disconnected one-factor branches within the observations used by the test. It need not hold outside that support.

Measurement and construction

Table 8 is the construction map from the paper’s notation to the Compustat and CRSP construction. It also records the capital and outcome timing, income boundaries, dollar scaling, and empirical event definition used throughout the paper.

Table 8

Measurement, definitions, and timing. The table maps the paper’s notation to the Compustat and CRSP construction. Dollar quantities are measured as shares of nominal GDP expressed in a 2019-sized economy.

<i>Panel A. Sample and timing</i>		
Object	Construction	Source, screen, or timing
Analysis sample	Annual CRSP/Compustat public operating firms, 1971–2019; 142,480 firm-years for 13,904 firms	Excludes financials, utilities, government-related firms, legacy data-quality failures, acquisitions above 20 percent of assets, and firms below the size and employment screens
Dollar units	$\tilde{X}_t = X_t \text{GDP}_{2019} / \text{GDP}_t$	Applied before accounting identities; removes aggregate nominal growth while preserving within-year ratios
Installed capital	Model K_t corresponds to empirical K_{t-1}	Exact preceding fiscal year for the same firm
One-year outcomes	Outcome at $t + 1$ is attached to the state observed at t	Exact next fiscal year for the same firm
Adverse CRSP exit	Terminal public-panel row matched through CCM to CRSP DLSTCD 4xx or 5xx	Generic disappearance and M&A are not events
<i>Panel B. Accounting primitives</i>		
Revenue	$R_t = SL_t = \text{sale}_t$	$R_t > 0$ where logs are used
Capital, depreciation, and investment	$K_t = KT_t = \text{at}_t$, $DP_t = \text{dp}_t$, and $I_t = K_t - K_{t-1} + DP_t$	Capital is measured at fiscal-year end; investment covers the current fiscal interval
Claims and firm value	$EQ_t = \text{csho}_t \times \text{prcc}_t$, $DB_t = \text{lt}_t$, and $V_t = EQ_t + DB_t$	Fiscal-year end
Distributions	$DE_t = \text{dvt}_t + \text{prstk}_t - \text{sstk}_t$, $DD_t = \text{xint}_t + (DB_{t-1} - DB_t)$, and $DI_t = DE_t + DD_t$	Net distributions to equity and debt during the current fiscal interval
<i>Panel C. Income boundaries and coordinates</i>		
Gross income	$C_t^G = \text{cogs}_t$ and $\Pi_t^G = R_t - C_t^G$	$R_t, C_t^G > 0$ for coordinates
Operating income	$C_t^O = \text{cogs}_t + \text{xsga}_t$ and $\Pi_t^O = R_t - C_t^O$	$R_t, C_t^O > 0$ for coordinates
Economic income	$\Pi_t^E = I_t + DI_t$ and $C_t^E = R_t - \Pi_t^E$	Capital retention plus disbursement to stakeholders; $R_t, C_t^E > 0$ for coordinates
Scale and balance	$\lambda_t^j = (\log R_t + \log C_t^j)/2$ and $\tau_t^j = \log R_t - \log C_t^j$	Boundary-specific; break-even is $\tau_t^j = 0$
Scale intensity and income	$x_t^j = \lambda_t^j - \log K_{t-1}$ and $\Pi_t^j = 2 \exp(\lambda_t^j) \sinh(\tau_t^j/2)$	Beginning-capital coordinate; identity verified observation by observation
<i>Panel D. Corporate outcomes</i>		
Investment and capital growth	I_t/K_{t-1} and $\log K_t - \log K_{t-1}$	Current fiscal interval over beginning capital
Income-to-capital	Π_t/K_{t-1}	Current flow over beginning capital
Closing average q	V_t/K_t ; its exact lead is V_{t+1}/K_{t+1}	Value and capital are measured at the same fiscal-year end
Next loss	$1[\Pi_{t+1} < 0]$	Exact next fiscal year

Cross-boundary concordance and temporal aggregation

The accounting comparisons separately refit the surface at each income boundary and retain only common firm-years. Three-year means require three exactly consecutive observations.

Table 9
Cross-boundary concordance and temporal aggregation

<i>Panel A. Cross-boundary concordance</i>					
Boundary comparison	Firm-years	Pearson corr.	Rank corr.	Same sign (%)	Within-firm corr.
Operating-economic	142,480	0.765	0.774	82.4	0.619
Operating-gross	142,458	0.611	0.455	66.3	0.657
Economic-gross	142,458	0.501	0.376	63.2	0.449
<i>Panel B. Width under exact temporal aggregation</i>					
Boundary	Aggregation	Firm-years	Residual IQR	Balance IQR	Residual / balance
Operating	Annual	142,480	0.118	0.143	0.823
Operating	Three-year mean	109,928	0.103	0.132	0.781
Economic	Annual	142,480	0.098	0.115	0.855
Economic	Three-year mean	109,928	0.083	0.103	0.803
Gross	Annual	142,458	0.318	0.395	0.805
Gross	Three-year mean	109,908	0.298	0.368	0.809

Panel A compares separately cross-fitted residuals on common firm-years. Within-firm correlations subtract each firm's layer-specific median. Panel B uses exact consecutive three-year windows; the final column divides residual IQR by balance IQR. Each surface is re-estimated within its accounting boundary.

Operating- and economic-income residuals have rank correlation 0.774, agree in sign for 82.4 percent of common firm-years, and retain within-firm correlation 0.619. Under exact three-year means, residual IQR remains 0.781 and 0.803 of balance IQR at the operating- and economic-income boundaries. The balance residual therefore survives both accounting-boundary and temporal-aggregation changes.